

CLUSTER ALGEBRAS AND TRIANGULATED SURFACES

PART II: LAMBDA LENGTHS

SERGEY FOMIN AND DYLAN THURSTON

ABSTRACT. For any cluster algebra whose underlying combinatorial data can be encoded by a bordered surface with marked points, we construct a geometric realization in terms of suitable decorated Teichmüller space of the surface. On the geometric side, this requires opening the surface at each interior marked point into an additional geodesic boundary component. On the algebraic side, it relies on the notion of a non-normalized cluster algebra and the machinery of tropical lambda lengths.

Our model allows for an arbitrary choice of coefficients which translates into a choice of a family of integral laminations on the surface. It provides an intrinsic interpretation of cluster variables as renormalized lambda lengths of arcs on the surface. Exchange relations are written in terms of the shear coordinates of the laminations, and are interpreted as generalized Ptolemy relations for lambda lengths.

This approach gives alternative proofs for the main structural results from our previous paper, removing unnecessary assumptions on the surface.

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1. INTRODUCTION

Overview of the main results. This paper continues the study of cluster algebras associated with marked Riemann surfaces with holes and punctures. The emphasis of our first paper [9], written in collaboration with Michael Shapiro, was on the combinatorial construction of the cluster complex, which we showed to be closely related to the *tagged arc complex* of the surface, a particular extension of the classical simplicial complex of arcs connecting marked points. The focus of the current paper is on geometry, specifically on providing an explicit geometric interpretation for the cluster variables in *any* cluster algebra (of geometric type) whose exchange matrix can be associated with a triangulated surface. More concretely, we demonstrate that each cluster variable can be viewed as a properly normalized *lambda length* of the corresponding (tagged) arc. Introduced by R. Penner [26, 27, 28], lambda lengths serve as coordinates on appropriate *decorated Teichmüller spaces*; a point in such a space is a hyperbolic metric on a surface, together with some additional decoration. In fact, we use an extension of lambda lengths to *opened surfaces*, surfaces with some extra geodesic boundary.

The main underlying idea of this line of inquiry—already present in the pioneering works of V. Fock and A. Goncharov [5, 6], and of M. Gekhtman, M. Shapiro, and A. Vainshtein [21]—is to interpret a decorated Teichmüller space as the real positive part of an algebraic variety. The coordinate ring \mathcal{A} of this variety is generated by the variables corresponding to the lambda lengths; these variables satisfy certain algebraic relations among lambda lengths, known as generalized *Ptolemy relations*. The rings \mathcal{A} arising from various versions of this construction possess, in a very natural way, a cluster algebra structure: the lambda lengths become cluster variables, and clusters correspond to tagged triangulations.

The main results obtained in this paper can be succinctly summarized as follows.

- (1) We investigate a broad class of cluster algebras that includes any cluster algebra of geometric type whose (skew-symmetric) exchange matrix is a signed adjacency matrix of a triangulated bordered surface. Each such cluster algebra is naturally associated with a collection of integral *laminations* on the surface. Every exchange relation is then readily written in terms of *shear coordinates* of these laminations with respect to a given tagged triangulation.
- (2) We describe the cluster variables in these cluster algebras as generalized lambda lengths. To this end, we (a) extend the lambda length construction to tagged arcs, (b) define Teichmüller spaces of opened surfaces and the associated lambda lengths, (c) define *laminated Teichmüller spaces* and *tropical lambda lengths*, and (d) combine all these elements together to realize cluster variables as certain rescalings of lambda lengths of tagged arcs.
- (3) The underlying combinatorics of a cluster algebra is governed by its *cluster complex*. In our first paper [9], we described this complex in terms of tagged arcs, and investigated its basic structural properties. For technical reasons, some results in [9] required an exception that excluded closed surfaces with exactly two punctures. In this paper, we remove this restriction, extending the main results of [9] to arbitrary bordered surfaces with marked points.

Structure of the paper. Sections 2–4 are devoted to preliminaries on cluster algebras and exchange patterns. It turns out that the proper algebraic framework for our main construction is provided by the axiomatic setting of *non-normalized* cluster algebras. This setting, although the original one for the cluster algebra theory [12], was all but abandoned in the intervening years, as the main developments and applications dealt almost exclusively with normalized cluster algebras. Section 2 contains a review of non-normalized cluster algebras and the mutation rules used to define them. The only (minor) novelty here is Proposition 2.9. Section 3 discusses rescaling of cluster variables and the technicalities involved in constructing a normalized pattern by rescaling a non-normalized one. Section 4 recalls the notion of cluster algebras of geometric type, and discusses realizations of these algebras in which both cluster and coefficient variables are represented by positive functions on a topological space, in anticipation of Teichmüller-theoretic applications.

In Section 5, we review the main constructions and some of the main results of the prequel [9] to this paper: ordinary and tagged arcs on a bordered surface with marked points; triangulations, flips, and arc complexes (both ordinary and tagged); associated exchange graphs; and signed adjacency matrices and their mutations.

In Section 6, we formulate the first batch of our results, which describe the structure of cluster algebras whose exchange matrices come from triangulated surfaces. (The proofs come much later.) In brief, we extend all main structural theorems of [9] to arbitrary surfaces, including closed surfaces with two marked points, not covered in [9] because of the exceptional nature of the fundamental groups of the corresponding exchange graphs.

In Sections 7–11, we develop the hyperbolic geometry tools required for our main construction. Section 7 presents Penner’s concept of lambda length, and its basic algebraic properties. In Section 8, we adapt this concept to the tagged setting, and write down the appropriate versions of Ptolemy relations. This enables us to describe, in Theorem 8.6, the main structural features of a particular class of exchange patterns whose coefficient variables come from boundary segments. (This class of cluster algebras already appeared in [5, 6, 21].) To handle general coefficient systems, we need another geometric idea, introduced in Section 9: the concept of an *opened surface* obtained by replacing interior marked points by geodesic circular boundary components. Section 10 presents the corresponding version of lambda lengths (for the *lifted arcs* on the opened surface), and an appropriate variation of the decorated Teichmüller space. In Section 11, we show that suitably rescaled lambda lengths of lifted arcs form a non-normalized exchange pattern, providing a crucial building block for our main construction (to be completed in Section 15).

Sections 12–14 are devoted to the combinatorics of general coefficient systems of geometric type (equivalently, extended exchange matrices). As noted by Fock and Goncharov [7], W. Thurston’s shear coordinates for simple closed curves transform under ordinary flips according to the rules of matrix mutations. In Section 12, we review this beautiful theory; in Section 13, we extend its main results, namely Thurston’s coordinatization theorem for integral laminations and the matrix mutation rule, to the tagged setting. Section 14 discusses the notion of *tropical lambda length*, a

discrete analogue of Penner’s concept, in which hyperbolic lengths are replaced by a combination of transverse measures with respect to a family of laminations. Tropical lambda lengths satisfy the tropical version of exchange relations, which makes them ideally suited for the role of rescaling factors in our main construction.

This construction is presented in Section 15. Here we introduce the notion of a *laminated Teichmüller space* whose defining data include, in addition to the surface, a fixed multi-lamination on it. This space can be coordinatized by yet another version of lambda lengths, obtained by dividing the ordinary lambda lengths of lifted arcs by the tropical ones. (The ratio does not depend on the choice of a lift of the arc. Although it does depend on the choice of lifted laminations, this choice does not affect the resulting cluster algebra structure, up to a canonical isomorphism.) These “laminated lambda lengths” form an exchange pattern (thus generate a cluster algebra) of the required kind. In other words, they satisfy the exchange relations of Ptolemy type whose coefficients are encoded by the shear coordinates of the chosen multi-lamination with respect to the current tagged triangulation. This geometric realization allows us to prove all our claims, made in earlier sections, pertaining to the structural properties of the cluster algebras under consideration.

The next two sections are devoted to applications and examples. In Section 16, we provide topological models for cluster-algebraic structures in coordinate rings of various algebraic varieties, such as certain Grassmannians and affine base spaces. In each case, the relevant cluster structure is encoded by a particular choice of a bordered surface with a collection of integral laminations on it. These examples provide fascinating links—awaiting further exploration—between representation theory and combinatorial topology. Section 17 treats in concrete detail two important classes of coefficient systems introduced in [14]: the *principal coefficients* and the *universal coefficients* (the latter in finite types A_n and D_n only).

Appendix A contains an informal discussion of how our main geometric construction of renormalized lambda lengths on opened surfaces can be obtained by means of *tropical degeneration* from a somewhat more conventional setting in which, instead of fixing a multi-lamination, one picks in advance a decorated hyperbolic structure on an opened surface.

Appendix B is designed to help the reader navigate between the various versions of Teichmüller spaces and their respective coordinatizations.

Our notation and terminology agree with the previous paper in the series [9].

Several figures in this paper are best viewed in **color**.

Historical note. A preliminary version of this paper (64 pages long) was circulated in May 2008, and posted on our respective web sites. While that version contained all the main results and proofs, we were not satisfied with the exposition and felt the need to double-check the details of our setup. Indeed, careful inspection revealed a number of flaws, each of them fixable. In particular, the factor $\nu(p)$ in Definition 10.11 used to be different, which meant that the older version of Lemma 10.15 did not work uniformly for plain and notched arcs. Similarly, Definition 14.4 for notched arcs was missing the term $|l_{\overline{L}}(p)|$, which meant that Lemma 14.11 was not uniform, causing problems for the cluster algebra structure.

Unfortunately, it took us much longer than anticipated to complete the revision. We thank the friends and colleagues who kept up the pressure, urging us to finish the job.

We do not attempt to give an account of related developments in cluster algebra theory that took place after the preliminary version of this paper was circulated. In particular, we do not cite the papers referencing this work, except for mentioning a very brief and informal survey [18].

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2. NON-NORMALIZED CLUSTER ALGEBRAS

The original definition and basic properties of (non-normalized) cluster algebras were given in [12]; see [2] for further developments. In this section, we recall the basic notions of this theory following the aforementioned sources (cf. especially [2, Sections 1.1–1.2]). Note that our previous paper [9] used a more restrictive normalized setup.

The construction of a (non-normalized, skew-symmetrizable) cluster algebra begins with a *coefficient group* \mathbb{P} , an abelian group without torsion, written multiplicatively. Take the integer group ring $\mathbb{Z}\mathbb{P}$, and let \mathcal{F} be (isomorphic to) a field of rational functions in n independent variables with coefficients in $\mathbb{Z}\mathbb{P}$. Later, we are going to call the positive integer n the *rank* of our yet-to-be-defined cluster algebra, and \mathcal{F} its *ambient field*. The following definitions are central for the cluster algebra theory.

Definition 2.1 (*Seeds*). A *seed* in \mathcal{F} is a triple $\Sigma = (\mathbf{x}, \mathbf{p}, B)$ consisting of:

- a *cluster* $\mathbf{x} \subset \mathcal{F}$, a set of n algebraically independent elements (called *cluster variables*) which generate \mathcal{F} over the field of fractions of $\mathbb{Z}\mathbb{P}$;
- a *coefficient tuple* $\mathbf{p} = (p_x^\pm)_{x \in \mathbf{x}}$, a $2n$ -tuple of elements of \mathbb{P} ;
- an *exchange matrix* $B = (b_{xy})_{x,y \in \mathbf{x}}$, a *skew-symmetrizable* $n \times n$ integer matrix.

That is, B can be made skew-symmetric by rescaling its columns by appropriately chosen positive integer scalars. (In all applications in this paper, B will in fact be skew-symmetric.)

Definition 2.2 (*Seed mutations*). Let $\Sigma = (\mathbf{x}, \mathbf{p}, B)$ be a seed in \mathcal{F} , as above. Pick a cluster variable $z \in \mathbf{x}$. We say that another seed $\bar{\Sigma} = (\bar{\mathbf{x}}, \bar{\mathbf{p}}, \bar{B})$ is related to Σ by a *seed mutation* in direction z if

- the cluster $\bar{\mathbf{x}}$ is given by $\bar{\mathbf{x}} = \mathbf{x} - \{z\} \cup \{\bar{z}\}$, where the new cluster variable $\bar{z} \in \mathcal{F}$ is determined by the *exchange relation*

$$(2.1) \quad z\bar{z} = p_z^+ \prod_{\substack{x \in \mathbf{x} \\ b_{xz} > 0}} x^{b_{xz}} + p_z^- \prod_{\substack{x \in \mathbf{x} \\ b_{xz} < 0}} x^{-b_{xz}};$$

- the coefficient tuple $\bar{\mathbf{p}} = (\bar{p}_y^\pm)_{y \in \bar{\mathbf{x}}}$ satisfies

$$(2.2) \quad \bar{p}_{\bar{z}}^\pm = p_z^\mp;$$

$$(2.3) \quad \frac{\bar{p}_y^+}{\bar{p}_y^-} = \begin{cases} (p_z^+)^{b_{zy}} \frac{p_y^+}{p_y^-} & \text{if } b_{zy} \geq 0 \\ (p_z^-)^{b_{zy}} \frac{p_y^+}{p_y^-} & \text{if } b_{zy} \leq 0 \end{cases} \quad (\text{for } y \neq z)$$

- the exchange matrix $\bar{B} = (\bar{b}_{xy})$ is obtained from B by the *matrix mutation* rule

$$(2.4) \quad \bar{b}_{xy} = \begin{cases} 0 & \text{if } x = y = \bar{z}; \\ -b_{xz} & \text{if } y = \bar{z} \neq x; \\ -b_{zy} & \text{if } x = \bar{z} \neq y; \\ b_{xy} & \text{if } x \neq \bar{z}, y \neq \bar{z}, \text{ and } b_{xz}b_{zy} \leq 0; \\ b_{xy} + |b_{xz}|b_{zy} & \text{if } x \neq \bar{z}, y \neq \bar{z}, \text{ and } b_{xz}b_{zy} \geq 0. \end{cases}$$

It is easy to check that the mutation rule is symmetric: Σ is in turn related to $\bar{\Sigma}$ by a mutation in direction \bar{z} .

Remark 2.3. The crucial—and only—difference between Definition 2.2 and its counterpart [9, Definition 5.1] used in our previous paper is that here we do not require \mathbb{P} to be endowed with an “auxiliary addition” making it into a semifield (cf. Definition 3.3), and consequently eliminate the normalization requirement (cf. (3.2)). As noted in [12], there is a price to be paid for getting rid of the semifield structure: in the absence of normalization, the seed $\bar{\Sigma}$ in Definition 2.2 is not determined by Σ and a choice of z . Indeed, the $n - 1$ monomial formulas (2.3) prescribe the ratios \bar{p}_y^+/\bar{p}_y^- only, leaving us with $n - 1$ degrees of freedom in choosing specific coefficients \bar{p}_y^\pm .

Remark 2.4. The matrix mutation rule (2.4) has many equivalent reformulations; see, e.g., [12, (4.3)], [14, (2.2)], and [14, (2.5)]. In particular, the last two cases in (2.4) (where we assume that $x \neq \bar{z}$ and $y \neq \bar{z}$) can be restated as

$$(2.5) \quad \bar{b}_{xy} = b_{xy} + [b_{zy}]_+ b_{xz} + b_{zy} [-b_{xz}]_+$$

where we use the notation $[b]_+ = \max(b, 0)$. Similarly, (2.3) can be rewritten as

$$(2.6) \quad \frac{\bar{p}_y^+}{\bar{p}_y^-} = \frac{p_y^+}{p_y^-} \left(\frac{p_z^+}{p_z^-} \right)^{[b_{zy}]_+} (p_z^-)^{b_{zy}}.$$

Example 2.5 (*Seed mutation in rank 2*, cf. [12, Example 2.5]). For $n = 2$, the mutation rules simplify considerably. Let $\mathbf{x} = \{z, x\}$, $\mathbf{p} = (p_z^+, p_z^-, p_x^+, p_x^-)$, and

$$B = \begin{bmatrix} 0 & b_{zx} \\ b_{xz} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix} \quad (b, c > 0).$$

Performing mutation in direction z , we obtain:

- the cluster $\bar{\mathbf{x}} = \{\bar{z}, x\}$, with \bar{z} determined by the exchange relation

$$(2.7) \quad z \bar{z} = p_z^+ x^c + p_z^-;$$

- the coefficient tuple $\bar{\mathbf{p}} = (\bar{p}_z^+, \bar{p}_z^-, \bar{p}_x^+, \bar{p}_x^-)$ satisfying $\bar{p}_z^\pm = p_z^\mp$ and

$$(2.8) \quad \frac{\bar{p}_x^+}{\bar{p}_x^-} = (p_z^-)^{-b} \frac{p_x^+}{p_x^-};$$

- the exchange matrix

$$\bar{B} = \begin{bmatrix} 0 & \bar{b}_{z\bar{x}} \\ \bar{b}_{x\bar{z}} & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}.$$

Definition 2.6 (*Exchange pattern*). Let \mathbf{E} be a connected (unoriented, possibly infinite) n -regular graph. That is, each vertex t in \mathbf{E} is connected to exactly n other vertices; the corresponding n -element set of edges is denoted by $\text{star}(t)$. An *attachment* of a seed $\Sigma = (\mathbf{x}, \mathbf{p}, B)$ at t is a bijective labeling of the n cluster variables $x \in \mathbf{x}$ (hence the associated coefficient pairs p_x^\pm , and the rows/columns of B) by the n edges in $\text{star}(t)$; cf. [13, Section 2.2]. With such a seed attachment, we typically use the natural notation

$$\begin{aligned} \mathbf{x} &= \mathbf{x}(t) = (x_e(t))_{e \in \text{star}(t)}, \\ \mathbf{p} &= \mathbf{p}(t) = (p_e^\pm(t))_{e \in \text{star}(t)}, & p_e^\pm(t) &= p_{x_e(t)}^\pm, \\ B &= B(t) = (b_{ef}(t))_{e, f \in \text{star}(t)}, & b_{ef}(t) &= b_{x_e(t), x_f(t)}. \end{aligned}$$

An *exchange pattern* on \mathbf{E} is, informally speaking, a collection of seeds attached at the vertices of \mathbf{E} and related to each other by the corresponding mutations. Let us now be precise. An exchange pattern $\mathcal{M} = (\Sigma_t)$ is a collection of seeds $\Sigma_t = (\mathbf{x}(t), \mathbf{p}(t), B(t))$ labeled by the vertices t in \mathbf{E} and satisfying the following condition. Consider an edge e in \mathbf{E} connecting two vertices t and \bar{t} . Let $\mathbf{x}(t) = (x_f(t))_{f \in \text{star}(t)}$ be the cluster at t , labeled using the seed attachment. Then the definition requires that the seed $\Sigma_{\bar{t}}$ is related to Σ_t by a mutation in the direction of $x_e(t)$. In particular, we have

$$\mathbf{x}(\bar{t}) = \mathbf{x}(t) - \{x_e(t)\} \cup \{x_e(\bar{t})\},$$

while the exchange relation (2.1) associated with the edge e takes the form

$$(2.9) \quad x_e(t) x_e(\bar{t}) = p_e^+(t) \prod_{\substack{f \in \text{star}(t) \\ b_{fe}(t) > 0}} x_f(t)^{b_{fe}(t)} + p_e^-(t) \prod_{\substack{f \in \text{star}(t) \\ b_{fe}(t) < 0}} x_f(t)^{-b_{fe}(t)}.$$

Since the adjacent clusters $\mathbf{x}(t) = (x_f(t))_{f \in \text{star}(t)}$ and $\mathbf{x}(\bar{t}) = (x_{\bar{f}}(\bar{t}))_{\bar{f} \in \text{star}(\bar{t})}$ coincide (as unlabeled sets) except for the replacement of $x_e(t)$ by $x_e(\bar{t})$, we have a bijection

$$\begin{aligned} \text{star}(t) \setminus \{e\} &\rightarrow \text{star}(\bar{t}) \setminus \{e\} \\ f &\mapsto \bar{f} \end{aligned}$$

(defined by $x_f(t) = x_{\bar{f}}(\bar{t})$) between the edges incident to t and \bar{t} , respectively. These bijections form a *discrete connection* (see, e.g., [22]) on \mathbf{E} that keeps track of the relabelings of each cluster variable. The corresponding equations (2.2)–(2.3) become

$$(2.10) \quad \bar{p}_e^\pm(\bar{t}) = p_e^\mp(t);$$

$$(2.11) \quad \frac{p_f^\pm(\bar{t})}{p_{\bar{f}}^\mp(\bar{t})} = \begin{cases} (p_e^+(t))^{b_{ef}(t)} \frac{p_f^+(t)}{p_{\bar{f}}^-(t)} & \text{if } b_{ef}(t) \geq 0 \\ (p_e^-(t))^{b_{ef}(t)} \frac{p_f^-(t)}{p_{\bar{f}}^+(t)} & \text{if } b_{ef}(t) \leq 0 \end{cases} \quad (\text{for } f \neq e).$$

The matrix mutation rule can be similarly rewritten in this notation.

We note that an exchange pattern on a regular graph \mathbf{E} can be canonically lifted to an exchange pattern on its universal cover, an n -regular tree \mathbb{T}_n , recovering the original definition in [12].

Example 2.7 (*Exchange pattern of rank 2*). We continue with Example 2.5. Figure 1 shows a fragment of a 2-regular graph \mathbf{E} . We attach the seeds $(\mathbf{x}, \mathbf{p}, B)$ and $(\bar{\mathbf{x}}, \bar{\mathbf{p}}, \bar{B})$ at the adjacent vertices t and \bar{t} , so that

$$\begin{aligned} \mathbf{x} = (z, x) = \mathbf{x}(t) = (x_e(t), x_f(t)), & \quad \bar{\mathbf{x}} = (\bar{z}, x) = \mathbf{x}(\bar{t}) = (x_e(\bar{t}), x_{\bar{f}}(\bar{t})), \\ \mathbf{p} = (p_z^\pm, p_x^\pm) = \mathbf{p}(t) = (p_e^\pm(t), p_f^\pm(t)), & \quad \bar{\mathbf{p}} = (\bar{p}_z^\pm, \bar{p}_x^\pm) = \mathbf{p}(\bar{t}) = (p_e^\pm(\bar{t}), p_{\bar{f}}^\pm(\bar{t})), \\ B = B(t) = \begin{bmatrix} 0 & b_{ef}(t) \\ b_{fe}(t) & 0 \end{bmatrix} = \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix}, & \quad \bar{B} = B(\bar{t}) = \begin{bmatrix} 0 & b_{e\bar{f}}(\bar{t}) \\ b_{\bar{f}e}(\bar{t}) & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix}. \end{aligned}$$

The corresponding exchange relation (2.7) becomes

$$(2.12) \quad x_e(t) x_e(\bar{t}) = p_e^+(t) x_f(t)^c + p_e^-(t),$$

while the relation (2.8) becomes

$$(2.13) \quad \frac{p_{\bar{f}}^+(\bar{t})}{p_{\bar{f}}^-(\bar{t})} = (p_e^-(t))^{-b} \frac{p_f^+(t)}{p_f^-(t)}.$$

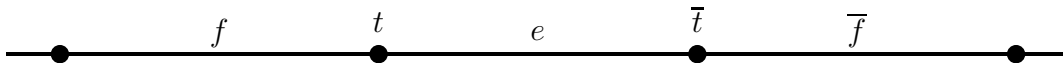


FIGURE 1. Exchange graph of rank 2

Definition 2.8 (*Cluster algebra*). To define a (*non-normalized*) cluster algebra \mathcal{A} , one needs two pieces of data:

- an exchange pattern $\mathcal{M} = (\Sigma_t)$ as in Definition 2.6;
- a *ground ring* \mathcal{R} , a subring with unit in $\mathbb{Z}\mathbb{P}$ that contains all coefficient tuples $\mathbf{p}(t)$, for all seeds $\Sigma_t = (\mathbf{x}(t), \mathbf{p}(t), B(t))$.

The cluster algebra \mathcal{A} is then defined as the \mathcal{R} -subalgebra of the ambient field \mathcal{F} generated by the union of all clusters $\mathbf{x}(t)$.

We conclude this section by showing that although the rules of non-normalized seed mutation do not determine the mutated seed uniquely (see Remark 2.3), certain expressions derived from each seed in a non-normalized exchange pattern form a discrete dynamical system, i.e., satisfy a self-contained set of mutation-like recurrences.

Proposition 2.9 below is an extension to the non-normalized case of an important observation already made in [6, Lemma 2.11] (for trivial coefficients), in [20, Lemma 1.3] (for tropical coefficients), and in [14, Proposition 3.9] (for arbitrary normalized coefficients). For a (non-normalized) exchange pattern as in Definition 2.6, let us denote

$$(2.14) \quad \hat{y}_e(t) = \frac{p_e^+(t)}{p_e^-(t)} \prod_{f \in \text{star}(t)} x_f(t)^{b_{fe}(t)}$$

(cf. [14, (3.7)]). Note that $\hat{y}_e(t)$ is nothing but the ratio of the two terms on the right-hand side of the exchange relation (2.9). Surprisingly, the n -tuple

$$\hat{\mathbf{y}}(t) = (\hat{y}_e(t))_{e \in \text{star}(t)}$$

uniquely determines all adjacent n -tuples $\hat{\mathbf{y}}(\bar{t})$:

Proposition 2.9. *Let e be an edge in \mathbf{E} connecting t and \bar{t} . Then*

$$(2.15) \quad \hat{y}_{\bar{f}}(\bar{t}) = \begin{cases} \hat{y}_e(t)^{-1} & \text{if } f = e; \\ \hat{y}_f(t) \hat{y}_e(t)^{[b_{ef}(t)]_+} (\hat{y}_e(t) + 1)^{-b_{ef}(t)} & \text{if } f \neq e. \end{cases}$$

In the language of [14], the equation (2.15) means that the quantities $\hat{y}_e(t)$ form a (normalized) *Y-pattern* in \mathcal{F} .

Proof. The case $f = e$ is immediate from (2.4) and (2.10). For $f \neq e$, we obtain:

$$\begin{aligned} \hat{y}_{\bar{f}}(\bar{t}) &= \frac{p_{\bar{f}}^+(\bar{t})}{p_{\bar{f}}^-(\bar{t})} \prod_{\bar{g} \in \text{star}(\bar{t})} x_{\bar{g}}(\bar{t})^{b_{\bar{g}\bar{f}}(\bar{t})} \\ (\text{by (2.14)}) &= \frac{p_f^+(t)}{p_f^-(t)} \left(\frac{p_e^+(t)}{p_e^-(t)} \right)^{[b_{ef}(t)]_+} \left(\frac{x_e(\bar{t})}{p_e^-(t)} \right)^{-b_{ef}(t)} \prod_{\bar{g} \neq e} x_g(t)^{b_{\bar{g}\bar{f}}(\bar{t})} \\ (\text{by (2.6)}) &= \hat{y}_f(t) \left(\prod_{g \neq e} x_g(t)^{-b_{gf}(t) + b_{\bar{g}\bar{f}}(\bar{t})} \right) \left(\frac{x_e(t)x_e(\bar{t})}{p_e^-(t)} \right)^{-b_{ef}(t)} \left(\frac{p_e^+(t)}{p_e^-(t)} \right)^{[b_{ef}(t)]_+} \end{aligned}$$

(by (2.14))

$$= \hat{y}_f(t) \left(\frac{p_e^+(t)}{p_e^-(t)} \prod x_g(t)^{b_{ge}(t)} \right)^{[b_{ef}(t)]_+} \left(\frac{x_e(t)x_e(\bar{t})}{p_e^-(t)} \prod x_g(t)^{-[-b_{ge}(t)]_+} \right)^{-b_{ef}(t)}$$

(by (2.5))

$$= \hat{y}_f(t) \hat{y}_e(t)^{[b_{ef}(t)]_+} (\hat{y}_e(t) + 1)^{-b_{ef}(t)}$$

(by (2.9) and (2.14)). \square

Remark 2.10. Equation (2.5) can now be recognized as, in some sense, a *tropical* version of Proposition 2.9. (This statement can be made precise using the construction analogous to the one introduced in Definition 4.1.)

Example 2.11. Continuing with the rank 2 case of Example 2.7, we get

$$(2.16) \quad \hat{y}_e(t) = \frac{p_e^+(t)}{p_e^-(t)} x_f(t)^{b_{fe}(t)} = \frac{p_e^+(t)}{p_e^-(t)} x_f(t)^c,$$

$$(2.17) \quad \hat{y}_f(t) = \frac{p_f^+(t)}{p_f^-(t)} x_e(t)^{b_{ef}(t)} = \frac{p_f^+(t)}{p_f^-(t)} x_e(t)^{-b},$$

$$(2.18) \quad \hat{y}_{\bar{f}}(\bar{t}) = \frac{p_{\bar{f}}^+(\bar{t})}{p_{\bar{f}}^-(\bar{t})} x_e(\bar{t})^{b_{e\bar{f}}(\bar{t})} = \frac{p_{\bar{f}}^+(\bar{t})}{p_{\bar{f}}^-(\bar{t})} x_e(\bar{t})^b.$$

Furthermore, (2.15) becomes

$$\hat{y}_{\bar{f}}(\bar{t}) = \hat{y}_f(t) (\hat{y}_e(t) + 1)^b,$$

which is straightforward to verify using (2.16)–(2.18), (2.12), and (2.13):

$$\hat{y}_{\bar{f}}(\bar{t}) = \frac{p_{\bar{f}}^+(\bar{t})}{p_{\bar{f}}^-(\bar{t})} x_e(\bar{t})^b = (p_e^-(t))^{-b} \frac{p_{\bar{f}}^+(t)}{p_{\bar{f}}^-(t)} \left(\frac{p_e^+(t) x_f(t)^c + p_e^-(t)}{x_e(t)} \right)^b = \hat{y}_f(t) (\hat{y}_e(t) + 1)^b.$$

3. RESCALING AND NORMALIZATION

In this section, we make a couple of observations related to rescaling of cluster variables; these observations will play an important role in the sequel.

The first algebraic observation (see Proposition 3.1 below) is that rescaling an exchange pattern gives again an exchange pattern. That is, if we replace each cluster variable by a new one that differs by a constant factor (these constant factors can be chosen completely arbitrarily for different cluster variables), and then rewrite each exchange relation in the obvious way in terms of the new variables, then the coefficients in these new exchange relations satisfy the monomial relations for an exchange pattern.

To formulate the above statement precisely, we will need the following natural notion. We say that a collection $(c_e(t))$ labeled by all pairs (t, e) with $e \in \text{star}(t)$ is *compatible with the discrete connection* defined by an exchange pattern \mathcal{M} (cf. Definition 2.6) if, for any edge e between t and \bar{t} , we have $c_f(t) = c_g(\bar{t})$ whenever $x_f(t) = x_g(\bar{t})$.

Proposition 3.1. *Let $\mathcal{M} = (\Sigma_t)$ be an exchange pattern on an n -regular graph \mathbf{E} , as in Definition 2.6. Let $(c_e(t))$ be a collection of scalars in \mathbb{P} that is compatible with the discrete connection associated with \mathcal{M} . Then the following construction yields an exchange pattern $\mathcal{M}' = (\Sigma'_t)$ on \mathbf{E} , with $\Sigma'_t = (\mathbf{x}'(t), \mathbf{p}'(t), B(t))$:*

- *the attached cluster $\mathbf{x}'(t) = (x'_e(t))_{e \in \text{star}(t)}$ is given by $x'_e(t) = x_e(t)/c_e(t)$;*
- *the coefficient tuple $\mathbf{p}'(t) = (p'_e(t))$ is defined by*

$$(3.1) \quad p'_e(t) = \frac{p_e^\pm(t)}{c_e(t)c_e(\bar{t})} \prod_{\pm b_{fe}(t) > 0} c_f(t)^{\pm b_{fe}(t)},$$

where \bar{t} denotes the endpoint of e different from t ;

- *the exchange matrices $B(t)$ do not change.*

Proof. It is straightforward to check that substituting $x_e(t) = x'_e(t)c_e(t)$ into (2.9) results into the requisite exchange relation in \mathcal{M}' . Together with the compatibility condition, this ensures that adjacent (attached) clusters are related by the corresponding mutations. It remains to demonstrate that the rescaled coefficient tuples $\mathbf{p}'(t)$ satisfy the requirements in Definition 2.2. The condition $p'_e(t) = p_e^\mp(\bar{t})$ (cf. (2.10)) is easily verified. Finally, in order to check the equation (2.11) for the coefficients $p'_e(t)$, we substitute the expressions (3.1) into it, and factor out (2.11) for the original pattern. The resulting equation

$$\prod_{\bar{g} \in \text{star}(\bar{t})} c_{\bar{g}}(\bar{t})^{b_{\bar{g}\bar{t}}(\bar{t})} = (c_e(t)c_e(\bar{t}))^{-b_{ef}(t)} \prod_{g: b_{ge}(t)b_{ef}(t) > 0} c_g(t)^{b_{ge}(t)|b_{ef}(t)|} \prod_{g \in \text{star}(t)} c_g(t)^{b_{gf}(t)}$$

is easily seen to follow from the matrix mutation rules (2.4). \square

Remark 3.2. For a more intuitive explanation of why the axioms of an exchange pattern survive rescaling, check this property against the alternative version of the mutation rules (2.2)–(2.4) given in [12, (2.7)] (cf. also the “Caterpillar Lemma” in [16]).

We next turn to the issue of *normalization*, that is, using the rescaling of cluster variables (as in Proposition 3.1) to obtain a “normalized” exchange pattern. The latter concept requires endowing the coefficient group \mathbb{P} with a semifield structure (cf. Remark 2.3).

Definition 3.3 (*Normalized exchange pattern*). Suppose that $(\mathbb{P}, \oplus, \cdot)$ is a (commutative) *semifield*, i.e., (\mathbb{P}, \cdot) is an abelian multiplicative group, (\mathbb{P}, \oplus) is a commutative semigroup, and the *auxiliary addition* \oplus is distributive with respect to the multiplication. (See Definition 4.1 for an example.) The multiplicative group of any such semifield \mathbb{P} is torsion-free [12, Section 5]. An exchange pattern $\mathcal{M} = (\Sigma_t)$ as in Definition 2.6 (or the corresponding cluster algebra) is called *normalized* if the coefficients $p_e^\pm(t)$ satisfy the normalization condition

$$(3.2) \quad p_e^+(t) \oplus p_e^-(t) = 1.$$

Our next algebraic observation is that rescaling of cluster variables in a non-normalized exchange pattern produces a normalized pattern if the rescaling factors $c_e(t)$ themselves satisfy the auxiliary-addition version of the same exchange relations.

Proposition 3.4. *Continuing with the assumptions and constructions of Proposition 3.1, let us furthermore suppose that the coefficient group \mathbb{P} is endowed with an additive operation \oplus making $(\mathbb{P}, \oplus, \cdot)$ a semifield. Then the rescaled pattern $\mathcal{M}' = (\Sigma'_t)$ is normalized if and only if the scalars $c_e(t)$ satisfy the relations*

$$(3.3) \quad c_e(t) c_e(\bar{t}) = p_e^+(t) \prod_{\substack{f \in \text{star}(t) \\ b_{fe}(t) > 0}} c_f(t)^{b_{fe}(t)} \oplus p_e^-(t) \prod_{\substack{f \in \text{star}(t) \\ b_{fe}(t) < 0}} c_f(t)^{-b_{fe}(t)}.$$

Proof. Equation (3.3) is simply a rewriting of the normalization condition $p_e^+(t) \oplus p_e^-(t) = 1$ for the rescaled coefficients $p_e^\pm(t)$ given by (3.1). \square

4. CLUSTER ALGEBRAS OF GEOMETRIC TYPE AND THEIR POSITIVE REALIZATIONS

The most important example of normalized exchange patterns (resp., normalized cluster algebras) are the patterns (resp., cluster algebras) of *geometric type*.

Definition 4.1 (*Tropical semifield, cluster algebra of geometric type* [12, Example 5.6, Definition 5.7]). Let I be a finite indexing set, and let

$$(4.1) \quad \mathbb{P} = \text{Trop}(q_i : i \in I)$$

be the multiplicative group of Laurent monomials in the formal variables $\{q_i : i \in I\}$, which we call the *coefficient variables*. Define the auxiliary addition \oplus by

$$(4.2) \quad \prod_i q_i^{a_i} \oplus \prod_i q_i^{b_i} = \prod_i q_i^{\min(a_i, b_i)}.$$

The semifield $(\mathbb{P}, \oplus, \cdot)$ is called a *tropical semifield* (cf. [1, Example 2.1.2]). A cluster algebra (or the corresponding exchange pattern) is said to be of *geometric type* if it is defined by a normalized exchange pattern with coefficients in some tropical semifield $\mathbb{P} = \text{Trop}(q_i : i \in I)$, over the ground ring $\mathcal{R} = \mathbb{Z}[q_i^\pm : i \in I]$ or $\mathcal{R} = \mathbb{Z}[q_i : i \in I]$.

This definition differs slightly from the one used in [2, 12, 13, 15], where only the former choice of \mathcal{R} was allowed. See *loc. cit.* for numerous examples.

Definition 4.2 (*Extended exchange matrix*). For an exchange pattern of geometric type, the coefficients $p_e^\pm(t)$ are monomials in the variables q_i . It is convenient and customary to encode these coefficients, along with the exchange matrix $B(t)$, in a rectangular *extended exchange matrix* $\tilde{B}(t) = (b_{ef}(t))$ defined as follows. The columns of $\tilde{B}(t)$ are, as before, labeled by $\text{star}(t)$. The top n rows of $\tilde{B}(t)$ are also labeled by $\text{star}(t)$ while the subsequent rows are labeled by the elements of I . The top $n \times n$ submatrix of $\tilde{B}(t)$ is $B(t)$ (so our notation for the matrix elements is consistent); the entries of the bottom $|I| \times n$ submatrix are uniquely determined by the formula

$$\frac{p_e^+(t)}{p_e^-(t)} = \prod_{i \in I} q_i^{b_{ie}(t)}.$$

As observed in [12], the mutation rules (2.10), (2.11), and (3.2) can be restated, in the case of geometric type, as saying that the matrices $\tilde{B}(t)$ undergo a matrix

mutation given by the same formulas (2.4) as before—now with a different set of row labels.

A pair $(\mathbf{x}(t), \tilde{B}(t))$ consisting of a cluster and the corresponding extended exchange matrix will be referred to as a *seed* (of geometric type).

Example 4.3 (*The ring $\mathbb{C}[\mathrm{SL}_2]$*). This was the first example given on the first page of the first paper about cluster algebras [12]. The coordinate ring

$$\mathcal{A} = \mathbb{C}[\mathrm{SL}_2] = \mathbb{C}[z_{11}, z_{12}, z_{21}, z_{22}] / \langle z_{11}z_{22} - z_{12}z_{21} - 1 \rangle$$

carries a structure of a cluster algebra of geometric type, of rank $n = 1$, with the coefficient semifield $\mathrm{Trop}(z_{12}, z_{21})$, the cluster variables z_{11} and z_{22} , and the sole exchange relation

$$z_{11}z_{22} = z_{12}z_{21} + 1.$$

The clusters are $\mathbf{x}(t_1) = \{z_{11}\}$ and $\mathbf{x}(t_2) = \{z_{22}\}$, and the extended exchange matrices are

$$\tilde{B}(t_1) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \tilde{B}(t_2) = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}.$$

Cf. Example 16.5.

The following concept is rooted in the original motivations of cluster algebras, designed in part to study totally positive parts of algebraic varieties of Lie-theoretic origin, in the sense of G. Lusztig [24] (see also [10, 15, 25] and references therein). In the context of cluster structures arising in Teichmüller theory, similar notions were first considered in [5, 21].

Definition 4.4 (*Positive realizations*). The positive realization of a cluster algebra \mathcal{A} of geometric type is, informally speaking, a faithful representation of \mathcal{A} in the space of positive real functions on a topological space \mathcal{T} of appropriate real dimension. An accurate definition follows.

Let \mathcal{A} be a cluster algebra of geometric type over the semifield $\mathrm{Trop}(q_i : i \in I)$. As before, let n denote the rank of \mathcal{A} , let \mathbf{E} be the underlying n -regular graph, and let $\mathbf{x}(t) = (x_e(t))_{e \in \mathrm{star}(t)}$, for $t \in \mathbf{E}$, be the clusters. A *positive realization* of \mathcal{A} is a topological space \mathcal{T} together with a collection of functions $x_e(t) : \mathcal{T} \rightarrow \mathbb{R}_{>0}$ and $q_i : \mathcal{T} \rightarrow \mathbb{R}_{>0}$ representing the cluster variables and the coefficient variables, respectively, so that

- these functions satisfy all appropriate exchange relations, and
- for each $t \in \mathbf{E}$, the map

$$(4.3) \quad \prod_{e \in \mathrm{star}(t)} x_e(t) \times \prod_{i \in I} q_i : \mathcal{T} \rightarrow \mathbb{R}_{>0}^{n+|I|}$$

is a homeomorphism.

The following simple observations will be useful in the sequel.

Proposition 4.5. *Every cluster algebra of geometric type has a positive realization, unique up to canonical homeomorphism.*

Conversely, let $(\tilde{B}(t))$ be a collection of extended exchange matrices $\tilde{B}(t)$ labeled by the vertices of an n -regular graph \mathbf{E} and related to each other by the corresponding matrix mutations (cf. Definition 4.2). Suppose furthermore that there exists a topological space \mathcal{T} and positive real functions $(x_e(t))$ and (q_i) on \mathcal{T} which satisfy the conditions in Definition 4.4. Then the matrices $(\tilde{B}(t))$ define a cluster algebra of geometric type, and the functions mentioned above provide its positive realization.

Proof. To construct a positive realization for a cluster algebra of geometric type, start with an arbitrary homeomorphism of the form (4.3) for one $t \in \mathbf{E}$; then determine the rest of the maps $x_e(t) : \mathcal{T} \rightarrow \mathbb{R}_{>0}$ using exchange relations. The key feature of exchange patterns that makes this construction work is that the re-parametrization maps relating adjacent clusters are birational and *subtraction-free* (hence positivity preserving).

For the second part, use the fact that two rational functions in m variables are equal if and only if they coincide pointwise as functions on $\mathbb{R}_{>0}^m$. \square

The fact that positive realizations are unique up to canonical homeomorphism allows us to speak of *the* positive realization.

5. BORDERED SURFACES, ARC COMPLEXES, AND TAGGED ARCS

This section offers a swift review of the main constructions in [9]. For a detailed exposition with lots of examples and pictures, see [9, Sections 2–5, 7].

Definition 5.1 (*Bordered surface with marked points*). Let \mathbf{S} be a connected oriented 2-dimensional Riemann surface with (possibly empty) boundary $\partial\mathbf{S}$. Fix a non-empty finite set \mathbf{M} of *marked points* in \mathbf{S} , so that there is at least one marked point on each connected component of $\partial\mathbf{S}$. Marked points in the interior of \mathbf{S} are called *punctures*. We will want \mathbf{S} to have at least one triangulation by a non-empty set of arcs with endpoints at \mathbf{M} . Consequently, we do not allow \mathbf{S} to be a sphere with one or two punctures; nor an unpunctured or once-punctured monogon; nor an unpunctured digon or triangle. We also exclude the case of a sphere with three punctures. Such a pair (\mathbf{S}, \mathbf{M}) is called a *bordered surface with marked points*. An example is shown in Figure 2.

Definition 5.2 (*Ordinary arcs*). An *arc* γ in (\mathbf{S}, \mathbf{M}) is a curve in \mathbf{S} , considered up to isotopy, such that

- the endpoints of γ are marked points in \mathbf{M} ;
- γ does not intersect itself, except that its endpoints may coincide;
- except for the endpoints, γ is disjoint from \mathbf{M} and from $\partial\mathbf{S}$; and
- γ does not cut out an unpunctured monogon or an unpunctured digon.

An arc whose endpoints coincide is called a *loop*. We denote by $\mathbf{A}^\circ(\mathbf{S}, \mathbf{M})$ the set of all arcs in (\mathbf{S}, \mathbf{M}) . See Figure 3.

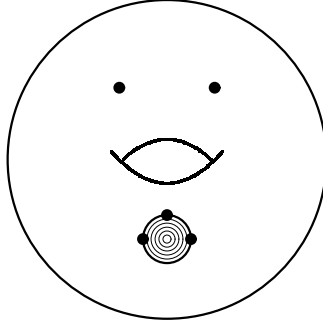


FIGURE 2. A bordered surface with marked points. In this example, \mathbf{S} is a torus with a hole; the boundary $\partial\mathbf{S}$ has a single component with 3 marked points on it; and the set \mathbf{M} consists of those 3 points plus 2 punctures in the interior of \mathbf{S} .

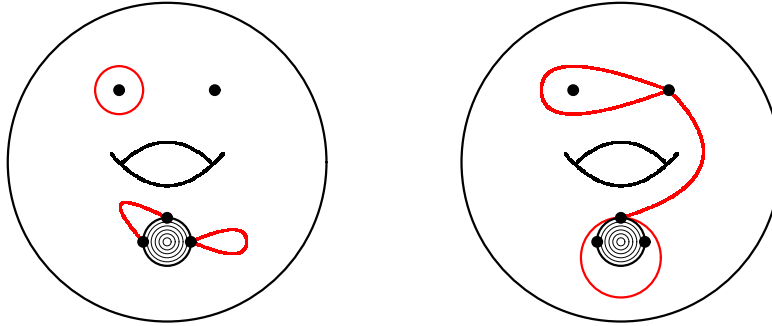


FIGURE 3. The three curves on the left do not represent arcs; the three curves on the right do.

Definition 5.3 (*Compatibility of ordinary arcs*). Two arcs are *compatible* if they (more precisely, some of their isotopic deformations) do not intersect in the interior of \mathbf{S} . For example, the three arcs shown in Figure 3 on the right are pairwise compatible.

Definition 5.4 (*Ideal triangulations*). A maximal collection of distinct pairwise compatible arcs forms an (ordinary) *ideal triangulation*. The arcs of a triangulation cut \mathbf{S} into *ideal triangles*; note that we do allow *self-folded* triangles, triangles where two sides are identified. Each ideal triangulation consists of

$$(5.1) \quad n = 6g + 3b + 3p + c - 6$$

arcs, where g is the genus of \mathbf{S} , b is the number of boundary components, p is the number of punctures, and c is the number of marked points on the boundary $\partial\mathbf{S}$.

Figure 4 shows two triangulations of a sphere with 4 punctures. Each triangulation has 4 ideal triangles, 2 of which are self-folded.

The assumptions made above ensure that (\mathbf{S}, \mathbf{M}) possesses a triangulation without self-folded triangles (see [9, Lemma 2.13]).



FIGURE 4. Two triangulations of a sphere with 4 punctures. In this example, $n = 6$, $g = 0$, $b = 0$, $p = 4$, $c = 0$, in the notation of (5.1).

Definition 5.5 (*Ordinary flips*). Ideal triangulations are connected with each other by sequences of *flips*. Each flip replaces a single arc γ in a triangulation T by a (unique) arc $\gamma' \neq \gamma$ that, together with the remaining arcs in T , forms a new ideal triangulation. See Figure 5; also, the two triangulations in Figure 4 are related by a flip.

It is important to note that this operation cannot be applied to an arc γ that lies inside a self-folded triangle in T .

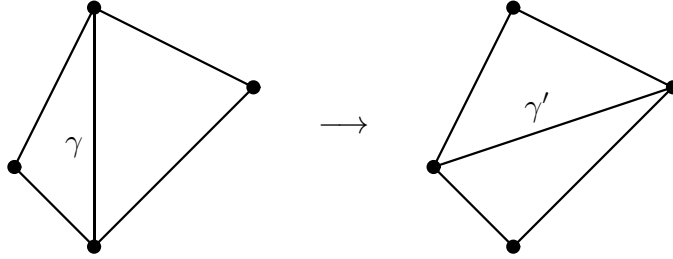


FIGURE 5. A flip inside a quadrilateral

Definition 5.6 (*Arc complex and its dual graph*). The *arc complex* $\Delta^\circ(\mathbf{S}, \mathbf{M})$ is the (possibly infinite) simplicial complex on the ground set $\mathbf{A}^\circ(\mathbf{S}, \mathbf{M})$ defined as the clique complex for the compatibility relation. In other words, the vertices of $\Delta^\circ(\mathbf{S}, \mathbf{M})$ are the arcs, and the maximal simplices are the ideal triangulations. The dual graph of $\Delta^\circ(\mathbf{S}, \mathbf{M})$ is denoted by $\mathbf{E}^\circ(\mathbf{S}, \mathbf{M})$; its vertices are the triangulations, and its edges correspond to the flips. See Figure 6.

In general, the arc complex $\Delta^\circ(\mathbf{S}, \mathbf{M})$ has nonempty boundary since its dual graph is not n -regular: not every arc can be flipped. In [9], we suggested a natural way to extend the arc complex beyond its boundary, obtaining an n -regular graph which can be used to build the desired exchange patterns. This requires the concept of a tagged arc.

Definition 5.7 (*Tagged arcs*). A *tagged arc* is obtained by taking an arc that does not cut out a once-punctured monogon and marking (“tagging”) each of its ends in one of the two ways, *plain* or *notched*, so that the following conditions are satisfied:

- an endpoint lying on the boundary of \mathbf{S} must be tagged plain, and
- both ends of a loop must be tagged in the same way.

See Figure 7.

The set of all tagged arcs in (\mathbf{S}, \mathbf{M}) is denoted by $\mathbf{A}^\times(\mathbf{S}, \mathbf{M})$.

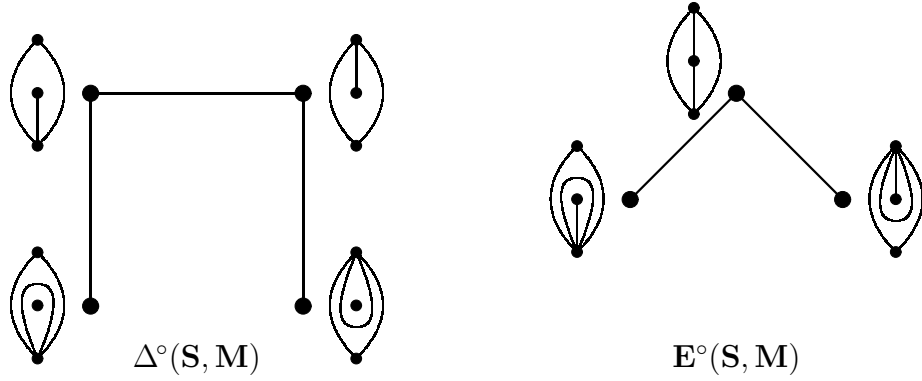


FIGURE 6. The arc complex and its dual graph for a once-punctured digon

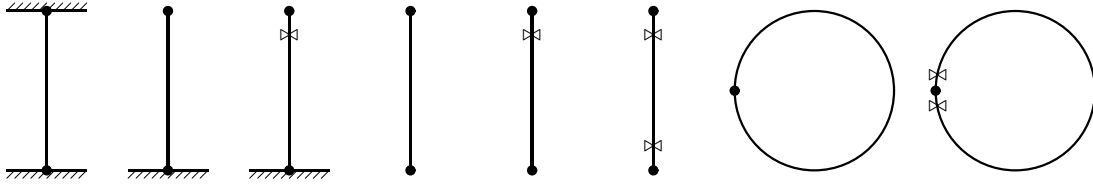


FIGURE 7. Different types of tagged arcs

Definition 5.8 (*Representing ordinary arcs by tagged arcs*). Ordinary arcs can be viewed as a special case of tagged arcs, via the following dictionary. Let us canonically represent any ordinary (untagged) arc β by a tagged arc $\tau(\beta)$ defined as follows. If β does not cut out a once-punctured monogon, then $\tau(\beta)$ is simply β with both ends tagged plain. Otherwise, β is a loop based at some marked point a and cutting out a punctured monogon with the sole puncture b inside it. Let α be the unique arc connecting a and b and compatible with β . Then $\tau(\beta)$ is obtained by tagging α plain at a and notched at b . See Figure 8.



FIGURE 8. Representing an arc bounding a punctured monogon by a tagged arc

Definition 5.9 (*Compatibility of tagged arcs*). This is an extension of the corresponding notion for ordinary arcs. Tagged arcs α and β are *compatible* if and only if

- their untagged versions α° and β° are compatible;
- if α and β share an endpoint a , then the ends of α and β connecting to a must be tagged in the same way—unless $\alpha^\circ = \beta^\circ$, in which case at least one end of α must be tagged in the same way as the corresponding end of β .

It is easy to see that the map $\gamma \mapsto \tau(\gamma)$ described in Definition 5.8 preserves compatibility.

Definition 5.10 (*Tagged triangulations*). A maximal (by inclusion) collection of pairwise compatible tagged arcs is called a *tagged triangulation*.

Each ideal triangulation T can be represented by a tagged triangulation $\tau(T)$ via the dictionary τ described above. Figure 9 shows two tagged triangulations obtained by applying τ to the triangulations in Figure 4.



FIGURE 9. Two tagged triangulations of a 4-punctured sphere.

All tagged triangulations have the same cardinality n given by (5.1) [9, Theorem 7.9].

Definition 5.11 (*Tagged arc complex*). The *tagged arc complex* $\Delta^\infty(\mathbf{S}, \mathbf{M})$ is the simplicial complex whose vertices are tagged arcs and whose simplices are collections of pairwise compatible tagged arcs. See Figure 10 on the left.

The ordinary arc complex $\Delta^\circ(\mathbf{S}, \mathbf{M})$ can be viewed a subcomplex of $\Delta^\infty(\mathbf{S}, \mathbf{M})$ (via the map τ); cf. Figures 6 and 10.

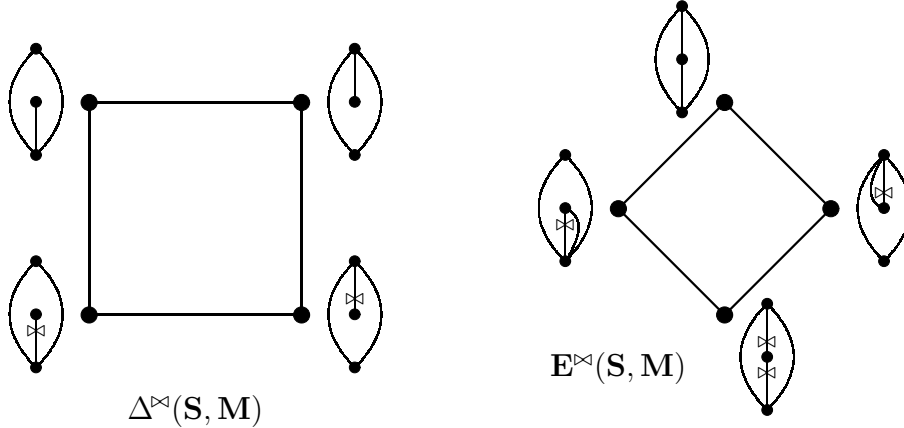


FIGURE 10. Tagged arc complex and its dual graph for a once-punctured digon

The maximal simplices of $\Delta^\infty(\mathbf{S}, \mathbf{M})$ are the tagged triangulations, so $\Delta^\infty(\mathbf{S}, \mathbf{M})$ is *pure* of dimension $n - 1$. Furthermore, $\Delta^\infty(\mathbf{S}, \mathbf{M})$ is a *pseudomanifold*, i.e., each simplex of codimension 1 is contained in precisely two maximal simplices. To rephrase, for every tagged arc in an arbitrary tagged triangulation, we can apply a *tagged flip* (replace it by a different tagged arc) in a unique way to produce another tagged triangulation.

Definition 5.12 (*Dual graph of the tagged arc complex*). The dual graph $\mathbf{E}^\infty(\mathbf{S}, \mathbf{M})$ of the pseudomanifold $\Delta^\infty(\mathbf{S}, \mathbf{M})$ has tagged triangulations as its vertices. Two such

vertices are connected by an edge if these tagged triangulations are related by a tagged flip. Thus, $\mathbf{E}^\bowtie(\mathbf{S}, \mathbf{M})$ is a (possibly infinite) n -regular graph. An example is shown in Figure 10 on the right.

Remark 5.13 (*Nomenclature of tagged triangulations and tagged flips*). Compatibility of tagged arcs is invariant with respect to a simultaneous change of all tags at a given puncture. Let us take any tagged triangulation T' and perform such a change at every puncture where all ends of T' are notched. It is not hard to verify that the resulting tagged triangulation T'' represents some ideal triangulation T (possibly containing self-folded triangles): $T'' = \tau(T)$. In other words, each tagged triangulation can be obtained from an ordinary one by applying τ and then placing notches on all arcs around some punctures.

This observation can be used to give a concrete description of all possible tagged flips. Let T' be a tagged triangulation, and let T'' and T be as above. Then each of the n tagged flips out of T' is of one of the two kinds:

- (5.D) a flip performed inside a once-punctured digon, as represented by one of the 4 edges of the graph $\mathbf{E}^\bowtie(\mathbf{S}, \mathbf{M})$ in Figure 10. The tagging at the vertices of the digon does not change; or
- (5.Q) an ordinary flip inside a quadrilateral in T (cf. Definition 5.5). As before, the sides of the quadrilateral do not have to be distinct. Moreover, those sides (stripped of their tagging) should be arcs of T but not necessarily of T' : specifically, such a side can be a loop in T enclosing a once-punctured monogon. The tagging at each vertex of the quadrilateral remains the same.

See [9, Section 9.2] for more discussion and proofs. For example, the two tagged triangulations in Figure 9 are related by a tagged flip of type (5.Q).

By [9, Proposition 7.10], the dual graph $\mathbf{E}^\bowtie(\mathbf{S}, \mathbf{M})$ (hence the complex $\Delta^\bowtie(\mathbf{S}, \mathbf{M})$) is connected—i.e., any two tagged triangulations can be connected by a sequence of flips—unless (\mathbf{S}, \mathbf{M}) is a surface with no boundary and a single puncture, in which case $\mathbf{E}^\bowtie(\mathbf{S}, \mathbf{M})$ (resp., $\Delta^\bowtie(\mathbf{S}, \mathbf{M})$) consists of two isomorphic connected components, one in which the ends of all tagged arcs are plain, and another in which they are all notched.

Definition 5.14 (*Exchange graph of tagged triangulations*). We denote by $\mathbf{E}(\mathbf{S}, \mathbf{M})$ a connected component of $\mathbf{E}^\bowtie(\mathbf{S}, \mathbf{M})$. More precisely, $\mathbf{E}(\mathbf{S}, \mathbf{M}) = \mathbf{E}^\bowtie(\mathbf{S}, \mathbf{M})$ unless (\mathbf{S}, \mathbf{M}) has no boundary and a single puncture; in the latter case, $\mathbf{E}(\mathbf{S}, \mathbf{M})$ is the connected component of $\mathbf{E}^\bowtie(\mathbf{S}, \mathbf{M})$ in which all arcs are plain.

As shown in [9, Theorem 7.11], there is a natural class of normalized exchange patterns (equivalently, cluster algebras) whose underlying graph is $\mathbf{E} = \mathbf{E}(\mathbf{S}, \mathbf{M})$. (Strictly speaking, this result was obtained under the assumption that (\mathbf{S}, \mathbf{M}) is not a closed surface with two punctures. We are not going to make this assumption herein.) In such a pattern, the cluster variables are labeled by the tagged arcs while clusters correspond to tagged triangulations. We review this construction.

Definition 5.15 (*Signed adjacency matrix*). The key ingredient in building an exchange pattern on $\mathbf{E}(\mathbf{S}, \mathbf{M})$ is a rule that associates with each tagged triangulation

T a skew-symmetric exchange matrix $B(T)$ called a *signed adjacency matrix* of T . The rows and columns of $B(T)$ are labeled by the arcs in T . The direct definition of $B(T)$ is fairly technical, and we refer the reader to [9, Definitions 4.1, 9.18] for those technicalities. For the immediate purposes of this review, we make the following shortcut. Let us start by defining $B(T)$ for an ordinary ideal triangulation T without self-folded triangles; this was first done in [5, 6, 21]. Under that assumption, one sets

$$(5.2) \quad B(T) = \sum_{\Delta} B^{\Delta},$$

the sum over all ideal triangles Δ in T of the $n \times n$ matrices $B^{\Delta} = (b_{ij}^{\Delta})$ given by

$$(5.3) \quad b_{ij}^{\Delta} = \begin{cases} 1 & \text{if } \Delta \text{ has sides } i \text{ and } j, \text{ with } j \text{ following } i \text{ in the clockwise order;} \\ -1 & \text{if the same holds, with the counterclockwise order;} \\ 0 & \text{otherwise.} \end{cases}$$

See Figure 11 for an example.

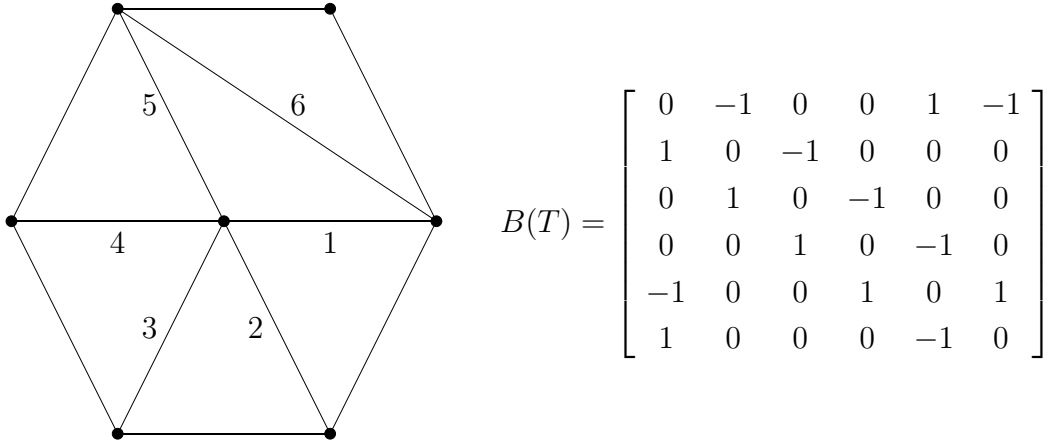


FIGURE 11. The signed adjacency matrix for a triangulation of a once-punctured hexagon

One can then extend the definition of $B(T)$ to arbitrary tagged triangulations by requiring that

- whenever T and \overline{T} are related by a flip of a tagged arc k , the associated signed adjacency matrices are related by the corresponding mutation:

$$(5.4) \quad B(\overline{T}) = \mu_k(B(T));$$

- if T is a triangulation without self-folded triangles, and T' is obtained from T by putting a notch at each end of each arc in T adjacent to a puncture, then $B(T') = B(T)$.

It follows from [9, Proposition 7.10, Definition 9.6, Lemma 9.7] that this definition is consistent, that is, there exists a unique collection of matrices $B(T)$ satisfying (5.4).

Each matrix $B(T)$ is skew-symmetric, with entries equal to 0, ± 1 , or ± 2 .

It turns out that replacing ideal triangulations by (more general) tagged triangulations does not extend the class of the associated exchange matrices $B(T)$ [9, Proposition 12.3]: each matrix $B(T)$ corresponding to a tagged triangulation is identical, up to simultaneous permutations of rows and columns, to a matrix corresponding to an ordinary ideal triangulation.

Remark 5.16. There are various ways to extend the matrices $B(T)$ to rectangular matrices $\tilde{B}(T)$ (cf. Definition 4.2), creating coefficient systems of geometric type. A very general construction of this kind will be discussed in Section 12. Here we briefly discuss an easy special case that was already pointed out in [5, 21].

Let $\mathbf{B}(\mathbf{S}, \mathbf{M})$ denote the set of boundary segments between adjacent marked points on $\partial\mathbf{S}$. Consider the tropical coefficient semifield

$$\mathbb{P} = \text{Trop}(q_\beta : \beta \in \mathbf{B}(\mathbf{S}, \mathbf{M}))$$

generated by the variables q_β labeled by such boundary segments. For an ideal triangulation T without self-folded triangles, define the $(|\mathbf{B}(\mathbf{S}, \mathbf{M})| + n) \times n$ matrices $\tilde{B}(T)$ by the same equations (5.2)–(5.3) as before, but with the understanding that the row index i can now be either an arc or a boundary segment. The matrices $\tilde{B}(T)$ still satisfy the mutation rule $\tilde{B}(\bar{T}) = \mu_k(\tilde{B}(T))$. This can be deduced from (5.4) by gluing a triangle on the other side of each boundary segment (thus making it into a legitimate arc), then “freezing” all these arcs (i.e., not allowing to flip them). The same argument allows us to extend the definition of $\tilde{B}(t)$ to arbitrary tagged triangulations, as was done for $B(t)$ ’s, resulting in a well defined tropical coefficient system.

6. STRUCTURAL RESULTS

In this section, we formulate those of our results whose statements do not require any references to Teichmüller theory or hyperbolic geometry—even though their proofs will rely on geometric arguments. These results concern structural properties of exchange patterns (or cluster algebras) whose exchange matrices can be described as signed adjacency matrices of triangulations of a bordered surface. Most crucially, we show that, for any choice of (normalized) coefficients, there is an exchange pattern on the n -regular graph $\mathbf{E} = \mathbf{E}(\mathbf{S}, \mathbf{M})$ (see Definition 5.14) whose exchange matrices are the signed adjacency matrices $B(T)$. More precisely, we have the following theorem.

Theorem 6.1. *Let T_\circ be a tagged triangulation consisting of n tagged arcs in (\mathbf{S}, \mathbf{M}) . Let $\Sigma_\circ = (\mathbf{x}(T_\circ), \mathbf{p}(T_\circ), B(T_\circ))$ be a (normalized) seed as in Definitions 2.1 and 3.3, that is:*

- $\mathbf{x}(T_\circ)$ is an n -tuple of formal variables labeled by the arcs in T_\circ ;
- $\mathbf{p}(T_\circ)$ is a $2n$ -tuple of elements of a semifield \mathbb{P} satisfying (3.2);
- $B(T_\circ)$ is the signed adjacency matrix of T_\circ .

Then there is a unique exchange pattern (Σ_T) on $\mathbf{E}(\mathbf{S}, \mathbf{M})$ such that $\Sigma_{T_\circ} = \Sigma_\circ$.

More precisely, let \mathcal{F} be the field of rational functions in the variables $\mathbf{x}(T_\circ)$ with coefficients in $\mathbb{Z}\mathbb{P}$. Then there exist unique elements $x_\gamma(T) \in \mathcal{F}$ and $p_\gamma^\pm(T) \in \mathbb{P}$

labeled by the tagged triangulations $T \in \mathbf{E}(\mathbf{S}, \mathbf{M})$ and the tagged arcs $\gamma \in T$ such that

- every triple $\Sigma_T = (\mathbf{x}(T), \mathbf{p}(T), B(T))$ is a seed, where $\mathbf{x}(T) = (x_\gamma(T))_{\gamma \in T}$, $\mathbf{p}(T) = (p_\gamma^\pm(T))_{\gamma \in T}$, and $B(T)$ is the signed adjacency matrix of T ;
- each cluster variable $x_\gamma = x_\gamma(T)$ does not depend on T ;
- if a tagged triangulation T' is obtained from T by flipping a tagged arc $\gamma \in T$, then $\Sigma_{T'}$ is obtained from Σ_T by the seed mutation replacing x_γ by $x_{\gamma'}$.

Furthermore, all cluster variables x_γ (hence all seeds Σ_T) are distinct.

In the terminology of [12, Section 7], the last statement means that $\mathbf{E}(\mathbf{S}, \mathbf{M})$ is the *exchange graph* of the exchange pattern (Σ_T) .

Theorem 6.1 implies that the structural results obtained in [9, Theorem 5.6] hold in full generality, for arbitrary bordered surfaces with marked points:

Corollary 6.2. *Let \mathcal{A} be a cluster algebra whose exchange matrices arise from triangulations of a surface (\mathbf{S}, \mathbf{M}) . Then each seed in \mathcal{A} is uniquely determined by its cluster; the cluster complex (see [9, Definition 5.4]) and the exchange graph \mathbf{E} of \mathcal{A} do not depend on the choice of coefficients in \mathcal{A} ; the seeds containing a given cluster variable form a connected subgraph of \mathbf{E} ; and several cluster variables appear together in the same cluster if and only if every pair among them does. The cluster complex is the complex of tagged arcs, as in [9, Theorem 7.11]; it is the clique complex for its 1-skeleton, and is a connected pseudomanifold.*

Theorem 6.1 and Corollary 6.2 are proved in Section 15, using results and constructions from the intervening sections. The proof is based on interpreting the cluster variables x_γ as generalized *lambda lengths*, which are particular functions on the appropriately defined extension of the Teichmüller space of (\mathbf{S}, \mathbf{M}) .

7. LAMBDA LENGTHS ON BORDERED SURFACES WITH PUNCTURES

The machinery of lambda lengths was introduced and developed by R. Penner [26, 28] in his study of decorated Teichmüller spaces. In this section, we adapt Penner's definitions to the case at hand, and give a couple of useful geometric lemmas.

Throughout the paper, (\mathbf{S}, \mathbf{M}) is a bordered surface with marked points as described at the beginning of Section 5. The (cusped) *Teichmüller space* $\mathcal{T}(\mathbf{S}, \mathbf{M})$ consists of all complete finite-area hyperbolic structures with constant curvature -1 on $\mathbf{S} \setminus \mathbf{M}$, with geodesic boundary at $\partial\mathbf{S} \setminus \mathbf{M}$, considered up to $\text{Diff}_0(\mathbf{S}, \mathbf{M})$, diffeomorphisms of \mathbf{S} fixing \mathbf{M} that are homotopic to the identity. (Thus there is a cusp at each point of \mathbf{M} .) Our assumptions on (\mathbf{S}, \mathbf{M}) guarantee that $\mathcal{T}(\mathbf{S}, \mathbf{M})$ is non-empty. In fact, it is a manifold of dimension $n - p = 6g + 3b + 2p + c - 6$ in the notation of Definition 5.4.

For a given hyperbolic structure in $\mathcal{T}(\mathbf{S}, \mathbf{M})$, each arc can be represented by a unique geodesic. Since there are cusps at the marked points, such a geodesic segment is of infinite length. So if we want to measure the “length” of a geodesic arc between two marked points, we need to renormalize. This is done as follows.

Definition 7.1 (*Decorated Teichmüller space* [26, 27, 28]). A point in a decorated Teichmüller space $\tilde{\mathcal{T}}(\mathbf{S}, \mathbf{M})$ is a hyperbolic structure as above together with a collection of horocycles h_p , one around each cusp corresponding to a marked point $p \in \mathbf{M}$.

Appropriately interpreted, a *horocycle* around a cusp p is the set of points at an equal distance from p : although the cusp is infinitely far away from any point in the surface, there is still a well-defined way to compare the distance to p from two different points in the surface. A horocycle can also be characterized as a curve perpendicular to every geodesic to p .

Remark 7.2. Our definition is a common generalization of those explicitly given by Penner in *loc. cit.*, as we simultaneously decorate both the punctures and the marked points on the boundary. The possibility of extending his theory to this generality was already mentioned by Penner [28, comments following Theorem 5.10].

Recall that $\mathbf{B}(\mathbf{S}, \mathbf{M})$ denotes the set of segments of the boundary $\partial\mathbf{S}$ between two adjacent marked points. The cardinality of $\mathbf{B}(\mathbf{S}, \mathbf{M})$ is thus equal to c , the number of marked points on $\partial\mathbf{S}$.

Definition 7.3 (*Lambda lengths* [26, 27, 28]). Fix $\sigma \in \tilde{\mathcal{T}}(\mathbf{S}, \mathbf{M})$. Let γ be an arc in $\mathbf{A}^\circ(\mathbf{S}, \mathbf{M})$, or a boundary segment in $\mathbf{B}(\mathbf{S}, \mathbf{M})$. We will use the notation γ_σ for the geodesic representative of γ (relative to σ).

Let $l(\gamma) = l_\sigma(\gamma)$ be the signed distance along γ_σ between the horocycles at either end of γ (positive if the two horocycles do not intersect, negative if they do intersect). The *lambda length* $\lambda(\gamma) = \lambda_\sigma(\gamma)$ of γ is defined by¹

$$(7.1) \quad \lambda(\gamma) = \exp(l(\gamma)/2).$$

Definition 7.3 can also be interpreted in terms of a certain dot product between two null vectors corresponding to the two endpoints. See [26] and Remark 16.2.

For a given $\gamma \in \mathbf{A}^\circ(\mathbf{S}, \mathbf{M}) \cup \mathbf{B}(\mathbf{S}, \mathbf{M})$, one can view the lambda length

$$\lambda(\gamma) : \sigma \mapsto \lambda_\sigma(\gamma)$$

as a function on the decorated Teichmüller space $\tilde{\mathcal{T}}(\mathbf{S}, \mathbf{M})$. Penner shows that such lambda lengths can be used to coordinatize $\tilde{\mathcal{T}}(\mathbf{S}, \mathbf{M})$, as follows.

Theorem 7.4. *For any triangulation T of (\mathbf{S}, \mathbf{M}) , the map*

$$\prod_{\gamma \in T \cup \mathbf{B}(\mathbf{S}, \mathbf{M})} \lambda(\gamma) : \tilde{\mathcal{T}}(\mathbf{S}, \mathbf{M}) \rightarrow \mathbb{R}_{>0}^{n+c}$$

is a homeomorphism.

(Recall from (5.1) that n is the total number of arcs in T , and c is the number of marked points on the boundary.)

¹This definition coincides with the one in [28, Section 4] (or [21]), and differs by a factor of $\sqrt{2}$ from the definition in [26]. The choice made here makes Lemma 7.9 below work with no factors.

Remark 7.5. The first version of this theorem was proved by Penner [26, Theorem 3.1], which treats the case of closed surfaces with punctures. This was later extended [27, 28]; the most relevant statement for us is [28, Theorem 5.10], which is not quite the statement above, since there the punctures in the interior are not decorated by horocycles and therefore are treated differently. Theorem 7.4 follows easily from the same arguments, for instance using the doubling argument of [28, Theorem 5.10] to reduce it to the case of a closed surface.

For our purposes, the crucial property of lambda lengths is the “Ptolemy relation,” the basic prototype of an exchange relation in cluster algebras.

Proposition 7.6 (Ptolemy relations [26, Proposition 2.6(a)]). *Let*

$$\alpha, \beta, \gamma, \delta \in \mathbf{A}^\circ(\mathbf{S}, \mathbf{M}) \cup \mathbf{B}(\mathbf{S}, \mathbf{M})$$

be arcs or boundary segments (not necessarily distinct) that cut out a quadrilateral in \mathbf{S} ; we assume that the sides of the quadrilateral, listed in cyclic order, are $\alpha, \beta, \gamma, \delta$. Let η and θ be the two diagonals of this quadrilateral; see Figure 12. Then the corresponding lambda lengths satisfy the Ptolemy relation

$$(7.2) \quad \lambda(\eta)\lambda(\theta) = \lambda(\alpha)\lambda(\gamma) + \lambda(\beta)\lambda(\delta).$$

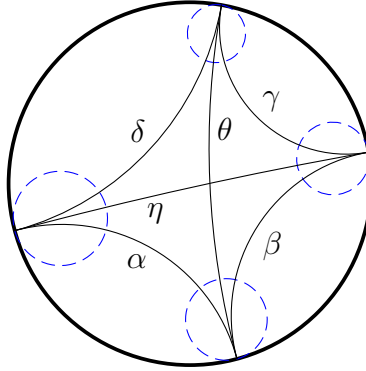


FIGURE 12. Sides and diagonals in a hyperbolic ideal quadrilateral, drawn in a lift to the hyperbolic plane

There is a Ptolemy relation (7.2) associated to each ordinary flip in an ideal triangulation (cf. Definition 5.5). Note that some sides of the relevant quadrilateral may be glued to each other, changing the appearance of the relation. See for example Figure 13.

By Theorem 7.4, each triangulation provides a set of coordinates on $\tilde{\mathcal{T}}(\mathbf{S}, \mathbf{M})$, while Proposition 7.6 allows us to relate the coordinatizations corresponding to different triangulations. In the absence of punctures, this leads to an exchange pattern (with a special choice of coefficients) in which the lambda lengths play the role of cluster variables; cf. [7, 21]. For a punctured surface, the situation is more delicate, for reasons both geometric and combinatorial: as we know, not every arc can be flipped without leaving the realm of ordinary triangulations. There is also a (related) algebraic reason, provided by the following lemma, a special case of Proposition 7.6.

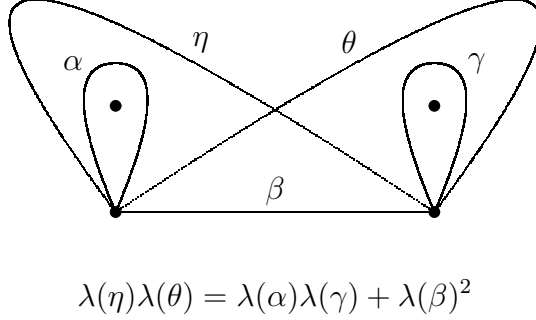


FIGURE 13. Ptolemy relation on a 4-punctured sphere. Cf. Figure 4.

Corollary 7.7. *Let $\alpha, \beta, \gamma, \eta, \theta \in \mathbf{A}^\circ(\mathbf{S}, \mathbf{M}) \cup \mathbf{B}(\mathbf{S}, \mathbf{M})$ be as shown in Figure 14, that is: α and β bound a digon with a sole puncture p inside it; θ and γ connect p to the vertices of the digon; η is the loop enclosing γ . Then*

$$(7.3) \quad \lambda(\eta)\lambda(\theta) = \lambda(\alpha)\lambda(\gamma) + \lambda(\beta)\lambda(\gamma).$$

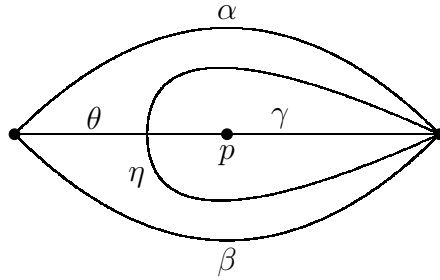


FIGURE 14. Arcs in a punctured digon.

The Ptolemy relation (7.3) cannot be an instance of a cluster exchange (2.1) since the two terms on the right-hand side of (7.3) have a common factor $\lambda(\gamma)$. Thus Corollary 7.7 shows that in the punctured case, complexities associated with setting up a cluster algebra structure already arise for ordinary flips, namely those that create self-folded triangles.

This issue can be resolved by introducing tagged arcs and their lambda lengths, and by extending the Ptolemy relations to the case of tagged flips. In the case of a tagged arc with a notched end, our definition of a lambda length will require the notion of (the hyperbolic distance from) a *conjugate horocycle*.

Definition 7.8. For an horocycle h around a puncture in the interior of \mathbf{S} , we denote by $L(h)$ the length of h as a (non-geodesic) curve in the hyperbolic metric. Two horocycles h and \bar{h} around the same interior marked point are called *conjugate* if $L(h)L(\bar{h}) = 1$.

Lemma 7.9 ([28, Lemma 4.4]). *Fix a decorated hyperbolic structure in $\tilde{\mathcal{T}}(\mathbf{S}, \mathbf{M})$. Consider a triangle in $\mathbf{S} \setminus \mathbf{M}$ with vertices $p, q, r \in \mathbf{M}$ whose sides have lambda lengths*

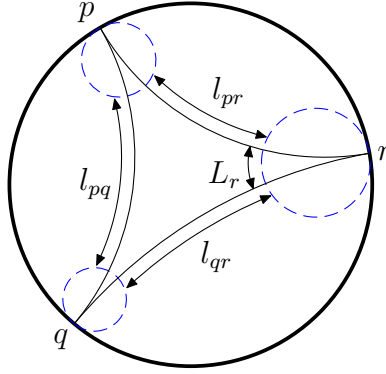


FIGURE 15. The lengths in the statement of Lemma 7.9.

λ_{pq} , λ_{pr} , and λ_{qr} . Then the length L_r of the horocyclic segment cut out by the triangle at vertex r is given by

$$L_r = \frac{\lambda_{pq}}{\lambda_{pr}\lambda_{qr}}.$$

Proof. First let us see how the side lengths l_{ij} , the lambda lengths λ_{ij} , and the horocyclic lengths L_i change when we change the choice of horocycle. For convenience, we work in the upper-half-plane model for the hyperbolic plane, with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2},$$

and put the three vertices at 0, 1, and ∞ , as in Figure 16 on the left. Let us move the horocycle around ∞ from an initial Euclidean height of y to a height of y' . By elementary integration, the new lengths are $L'_\infty = L_\infty(y/y')$, $l'_{i\infty} = l_{i\infty} + \ln(y'/y)$, and $\lambda'_{i\infty} = \lambda_{i\infty}\sqrt{y'/y}$, for $i \in \{0, 1\}$. The other lambda lengths and horocyclic lengths are unchanged. By symmetry, similar statements are true with $\{0, 1, \infty\}$ permuted.

From this we see that, up to scale, $\frac{\lambda_{01}}{\lambda_{0\infty}\lambda_{1\infty}}$ is the unique expression in the λ_{ij} that is covariant in the same way as L_∞ with respect to the $\mathbb{R}_{>0}^3$ -action associated with moving the three horocycles. To fix the scale, consider the case where all three horocycles just touch, as in Figure 16 on the right. In this case, $L_\infty = 1$, as in the statement of the lemma. \square

Lemma 7.9 appears as [28, Lemma 4.4]. A version of Lemma 7.9 also appears as [26, Corollary 3.4], but the statement there is off by a factor of $\sqrt{2}$.

Lemma 7.10. *Consider a punctured monogon with the vertex $q \in \partial\mathbf{S}$ and a sole puncture p in the interior. Choose a horocycle around q , and a horocycle h around p . Let λ_{qq} and λ_{pq} be the corresponding lambda lengths for the boundary of the monogon and the arc γ_{pq} connecting p and q inside it, respectively. Let \bar{h} be the horocycle around p conjugate to h , and let $\lambda_{\bar{p}q}$ be the corresponding lambda length of γ_{pq} ; see Figure 17. Then $\lambda_{qq} = \lambda_{pq}\lambda_{\bar{p}q}$.*

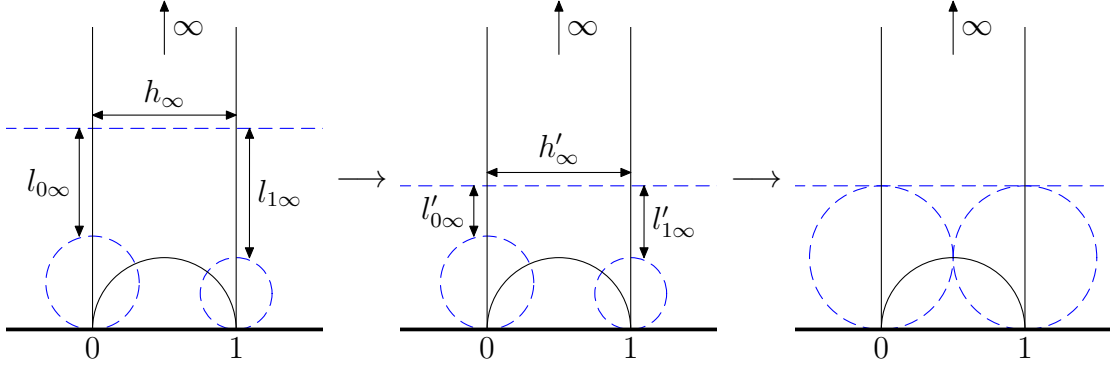


FIGURE 16. Moving a horocycle in the proof of Lemma 7.9. At the last step, all the hyperbolic lengths l_{xy} become 0.

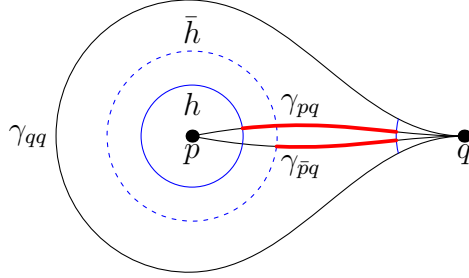


FIGURE 17. The punctured monogon in Lemma 7.10. The geodesic segments whose lengths are measured are shown in red.

Proof. By Lemma 7.9 applied to the self-folded triangle with sides γ_{pq} (twice) and γ_{qq} , we have $L(h) = \frac{\lambda_{qq}}{\lambda_{pq}^2}$; similarly, $L(\bar{h}) = \frac{\lambda_{qq}}{\lambda_{\bar{p}q}^2}$. Since h and \bar{h} are conjugate, we obtain $1 = L(h)L(\bar{h}) = \frac{\lambda_{qq}^2}{\lambda_{pq}^2 \lambda_{\bar{p}q}^2}$, and the claim follows. \square

8. LAMBDA LENGTHS OF TAGGED ARCS

Lemma 7.10 can be used to define a cluster algebra structure associated with the decorated Teichmüller space $\tilde{\mathcal{T}}(\mathbf{S}, \mathbf{M})$ of a general bordered surface with punctures, extending the construction in [5, 6, 21]. As mentioned earlier, the key idea is to interpret a notched end of a tagged arc as an indication that in defining the corresponding lambda length, we should take the distance to the *conjugate* horocycle.

Definition 8.1 (*Lambda lengths of tagged arcs*). Fix a decorated hyperbolic structure $\sigma \in \tilde{\mathcal{T}}(\mathbf{S}, \mathbf{M})$. The *lambda length* $\lambda(\gamma) = \lambda_\sigma(\gamma)$ of a tagged arc $\gamma \in \mathbf{A}^\infty(\mathbf{S}, \mathbf{M})$ is defined as follows. If both ends of γ are tagged plain, then the definition of $\lambda(\gamma)$

given in Definition 7.3 stands. Otherwise, the definition should be adjusted by replacing each horocycle h_p at a notched end p of γ by the corresponding conjugate horocycle \bar{h}_p .

In order to write relations among these lambda lengths, we will need the following lemma.

Lemma 8.2. *Let γ and γ' be two tagged arcs connecting marked points $p, q \in \mathbf{M}$. Assume that the untagged versions of γ and γ' coincide. Also assume that γ and γ' have identical tags at q , and different tags at p . (See Figure 18.) Let η be the loop based at q wrapping around p , so that η encloses a monogon with a sole puncture p inside it. Then $\lambda(\eta) = \lambda(\gamma)\lambda(\gamma')$, where we compute $\lambda(\eta)$ using h_q or \bar{h}_q according to whether γ and γ' are plain or notched at q , respectively.*

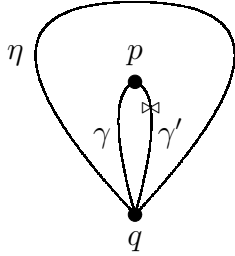


FIGURE 18. Arcs γ, γ' and the enclosing loop η in Lemma 8.2

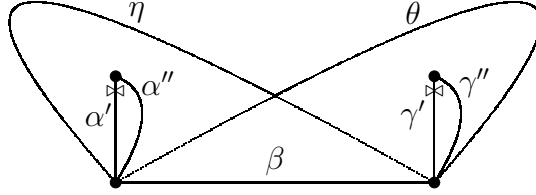
Proof. In view of Definition 8.1, Lemma 8.2 is a restatement of Lemma 7.10. \square

Remark 8.3. One delicate aspect associated with Lemma 8.2 is that η itself is *not* a legal tagged arc since it encloses a once-punctured monogon. Assume furthermore that both ends of η are tagged plain. Then η is an arc in $\mathbf{A}^\circ(\mathbf{S}, \mathbf{M})$; as such, it is represented by $\tau(\eta) = \gamma'$ (cf. Figure 8). However, $\lambda(\eta)$ is not the same as $\lambda(\gamma')$.

Lemma 8.2 allows us to write the exchange relations associated with the tagged flips of types (5.D) and (5.Q) described in Remark 5.13.

Definition 8.4 (*Ptolemy relations for tagged arcs*). If two tagged triangulations T_1 and T_2 are related by a flip of type (5.Q), then the corresponding lambda lengths are related by an appropriate specialization of the equation (7.2). We will continue to refer to such relations among lambda lengths of tagged arcs as (generalized) *Ptolemy relations*. Note that these relations can be more complicated than their counterparts for the ordinary arcs: some of the arcs $\alpha, \beta, \gamma, \delta$ appearing in (7.2) may bound a once-punctured monogon and so might not be present in T_1 and T_2 . In such a case, following Lemma 8.2 we should replace the lambda length of each such loop by the product of lambda lengths of the two tagged arcs in T_1 (equivalently, T_2) that it encloses. See Figure 19 for an example.

Definition 8.5 (*Digon relations*). For a tagged flip of type (5.D), consider the punctured digon as in Corollary 7.7 and Figure 14. Making use of Lemma 8.2, we



$$\lambda(\eta)\lambda(\theta) = \lambda(\alpha')\lambda(\alpha'')\lambda(\gamma')\lambda(\gamma'') + \lambda(\beta)^2$$

FIGURE 19. Ptolemy relation for a tagged flip in a 4-punctured sphere. Cf. Figures 9 and 13.

can rewrite the relation (7.3) in the form

$$(8.1) \quad \lambda(\gamma')\lambda(\theta) = \lambda(\alpha) + \lambda(\beta),$$

where γ' denotes γ with a notch at p , as in Figure 18. We refer to this as a *digon relation*. As in the case of Ptolemy relations, if one of the sides of the digon (α or β) bounds a punctured monogon, we replace the lambda length of the corresponding loop by the product of the lambda lengths of the two tagged arcs it encloses.

Our next goal is to show that lambda lengths of tagged arcs on a given bordered surface with marked points naturally form a normalized exchange pattern whose exchange relations are the relations of Definitions 8.4 and 8.5. Making these statements precise will require a bit of preparation.

First, the coefficient semifield \mathbb{P} is going to be the tropical semifield (see Definition 4.1) generated by the lambda lengths of the boundary segments (cf. Remark 5.16):

$$(8.2) \quad \mathbb{P} = \text{Trop}(\lambda(\gamma) : \gamma \in \mathbf{B}(\mathbf{S}, \mathbf{M})).$$

(If \mathbf{S} is closed, then $\mathbb{P} = \{1\}$ is the trivial one-element semifield.) We note that formula (8.2) makes sense in view of Theorem 7.4, which enables us to treat these lambda lengths as independent variables.

Second, let us describe the clusters. As in the case of ordinary arcs, each lambda length of a tagged arc $\gamma \in \mathbf{A}^\times(\mathbf{S}, \mathbf{M})$ can be viewed as a function $\sigma \mapsto \lambda_\sigma(\gamma)$ on the decorated Teichmüller space $\tilde{\mathcal{T}}(\mathbf{S}, \mathbf{M})$. For a tagged triangulation T of (\mathbf{S}, \mathbf{M}) , let

$$(8.3) \quad \mathbf{x}(T) = \{\lambda(\gamma) : \gamma \in T\}$$

denote the collection of lambda lengths of the tagged arcs in T .

Third, the ambient field. Let us pick an ordinary triangulation T_\circ without self-folded triangles. It follows from Theorem 7.4 that the lambda lengths in $\mathbf{x}(T_\circ)$ are algebraically independent over the field of fractions of \mathbb{P} . Let $\mathcal{F} = \mathcal{F}(T_\circ)$ be the field generated (say over \mathbb{R}) by these lambda lengths. That is, \mathcal{F} consists of all functions defined on a dense subset of $\tilde{\mathcal{T}}(\mathbf{S}, \mathbf{M})$ which can be written as a rational expression (say with real coefficients) in the lambda lengths of the arcs in T_\circ .

Theorem 8.6. *There exists a unique normalized exchange pattern (Σ_T) , here identified with its positive realization (see Definition 4.4 and Proposition 4.5) with the following properties:*

- *the coefficient semifield \mathbb{P} is the tropical semifield generated by the lambda lengths of boundary segments, as in (8.2);*
- *the ambient field $\mathcal{F} = \mathcal{F}(T_\circ)$ is generated over \mathbb{P} by the lambda lengths of a given triangulation T_\circ with no self-folded triangles;*
- *the underlying n -regular graph $\mathbf{E} = \mathbf{E}(\mathbf{S}, \mathbf{M})$ is the exchange graph of tagged triangulations (see Definition 5.14), and the seeds $\Sigma_T = (\mathbf{x}(T), \mathbf{p}(T), B(T))$ are labeled by the vertices of $\mathbf{E}(\mathbf{S}, \mathbf{M})$, as in Theorem 6.1;*
- *each cluster $\mathbf{x}(T)$ consists of the lambda lengths of the tagged arcs in T , as in (8.3);*
- *each exchange matrix $B(T)$ is the signed adjacency matrix of T ;*
- *the exchange relations out of each seed Σ_T are the Ptolemy relations (see Definition 8.4) and the digon relations (see Definition 8.5) associated with the two respective types of tagged flips from T (cf. Remark 5.13).*

Neither the ambient field $\mathcal{F}(T_\circ)$ nor the entire exchange pattern (Σ_T) depend on the choice of the initial triangulation T_\circ .

Proof. We know that the signed adjacency matrices $B(T)$ associated with tagged triangulations T satisfy the mutation rule (5.4), as required in the definition of an exchange pattern. We also know from Theorem 7.4 that the lambda lengths forming the initial cluster $\mathbf{x}(T_\circ)$ are algebraically independent. It remains to verify that

- (i) the relations (7.2) and (8.1) associated with arbitrary tagged flips can be viewed as exchange relations (2.9) for the signed adjacency matrices $B(T)$ of tagged triangulations T ,
- (ii) the coefficients appearing in these relations satisfy the mutation rules (2.10)–(2.11), and
- (iii) the normalization condition (3.2) holds in the tropical semifield \mathbb{P} .

Straightforward albeit somewhat tedious details of these verifications are omitted. It helps to note that a statement essentially equivalent to claim (ii) has been already checked in Remark 5.16. \square

Remark 8.7. It is tempting to try to deduce Theorem 6.1 from Theorem 8.6 by expressing cluster variables for any exchange pattern with exchange matrices $B(T)$ as lambda lengths of tagged arcs, perhaps rescaled to get different coefficients. It turns out however that this simplistic approach does not produce the most general coefficient patterns, as required for Theorem 6.1. Instead, we will need to develop, in subsequent sections, a more complicated concept of generalized lambda lengths for *laminated Teichmüller spaces* associated with *opened surfaces*.

Remark 8.8. We note that Theorem 8.6 implies that lambda lengths of tagged arcs in any cluster (i.e., tagged triangulation) parametrize the decorated Teichmüller space $\widetilde{\mathcal{T}}(\mathbf{S}, \mathbf{M})$, extending Theorem 7.4 verbatim to the case of tagged triangulations.

Example 8.9. Let (\mathbf{S}, \mathbf{M}) be a once-punctured digon, with notation as in Figure 14. The coefficient semifield is $\mathbb{P} = \text{Trop}(\lambda(\alpha), \lambda(\beta))$. The four cluster variables $\lambda(\gamma)$, $\lambda(\theta)$, $\lambda(\gamma')$, $\lambda(\theta')$, are labeled by the tagged arcs in (\mathbf{S}, \mathbf{M}) . The four clusters correspond to the four tagged triangulations, cf. Figure 10. The two exchange relations have the form (8.1). The resulting exchange pattern has finite type $A_1 \times A_1$, in the nomenclature of [13].

In the case of surfaces with no punctures, there is no tagging, and Theorem 8.6 specializes to its counterparts given by V. Fock and A. Goncharov [5, 6] and by M. Gekhtman, M. Shapiro, and A. Vainshtein [21]. The case of an unpunctured disk discussed in Example 8.10 below was already treated in [13, Section 12.2], without the hyperbolic geometry interpretation.

Example 8.10. Let (\mathbf{S}, \mathbf{M}) be an unpunctured $(n+3)$ -gon with vertices v_1, \dots, v_{n+3} , labeled counterclockwise. For $1 \leq i < j \leq n+3$, let γ_{ij} denote the arc or boundary segment connecting v_i and v_j , that is, a diagonal or a side of the $(n+3)$ -gon. Denote $\lambda_{ij} = \lambda(\gamma_{ij})$. Applying the construction in Theorem 8.6 to this special case, we get the coefficient semifield

$$\mathbb{P} = \text{Trop}(\lambda_{12}, \lambda_{23}, \dots, \lambda_{n+3,1})$$

generated by the lambda lengths of the sides of the $(n+3)$ -gon; the cluster variables are the lambda lengths of diagonals. The corresponding cluster algebra (of type A_n) can be interpreted (see [13, Proposition 12.7]) as a homogeneous coordinate ring of the Grassmannian $\text{Gr}_{2,n+3}$ of 2-dimensional subspaces in \mathbb{C}^{n+3} . See Example 16.1 for a more detailed treatment.

In the case of a once-punctured disk, we recover a particular cluster algebra of type D_n that has been described (from a different perspective) in [13, Section 12.4].

Example 8.11. Let (\mathbf{S}, \mathbf{M}) be an n -gon ($n \geq 3$) with vertices v_1, \dots, v_n (labeled counterclockwise) and a single puncture p inside it. For $1 \leq i < j \leq n$, there are two arcs or boundary segments connecting v_i and v_j , depending on which side of the curve the puncture p is on. Let γ_{ij} (resp., $\gamma_{i\bar{j}} = \gamma_{\bar{j}i}$) denote the curve that has p on the left (resp., right) as we move from v_i to v_j . There are also plain arcs $\gamma_{i\bar{i}}$ connecting v_i to p , and tagged arcs $\tilde{\gamma}_{i\bar{i}}$ that have a notched end at p . Replace γ 's with λ 's to denote the corresponding lambda lengths. Then $\lambda_{12}, \dots, \lambda_{n,1}$ generate the tropical semifield of coefficients; the remaining λ 's are cluster variables. As always, clusters correspond to tagged triangulations. The resulting cluster algebra coincides with the cluster algebra \mathcal{A}_\circ described in [13, Example 12.15], and identified in [13, Proposition 12.16] with the coordinate ring of the affine cone over the Schubert divisor in $\text{Gr}_{2,n+2}$.

9. OPENED SURFACES

As shown in Section 8, the lambda lengths of tagged arcs form an exchange pattern. It is important to note that the coefficients in such an exchange pattern are of a very special kind: they are monomials in the lambda lengths of the boundary segments. (For example, in the case of a closed surface with punctures, the coefficients are trivial.) In order to construct exchange patterns with general coefficients

(as in Theorem 6.1), we will need to modify our geometric setting, extending the Teichmüller space from surfaces with cusps at marked points to *opened surfaces*.

It is not unusual in Teichmüller theory to allow both cusped surfaces and surfaces with geodesic boundary in the same moduli and Teichmüller spaces. One standard model is the space of all complex structures on the complement of the marked points. A complex structure on the neighborhood of a singularity can have two possible behaviours: it can have a removable singularity at the marked point, corresponding to a cusp in the hyperbolic structure after uniformization; or it can be equivalent to the complex plane minus a closed disk, corresponding to a non-finite volume end after uniformization. It will be more convenient for us to use a different (equivalent) model: instead of complex structures, we will work with hyperbolic metrics, truncated so that they have geodesic boundary and finite volume. In addition, we add an orientation on each geodesic boundary component.

On the combinatorial/topological level, our construction will be based on the following concept.

Definition 9.1 (*Opening of a surface*). Let $\overline{\mathbf{M}} = \mathbf{M} \setminus \partial \mathbf{S}$ denote the set of punctures of \mathbf{S} . For a subset $P \subset \overline{\mathbf{M}}$, the corresponding *opened surface* \mathbf{S}_P is obtained from \mathbf{S} by removing a small open disk around each point in P . For $p \in P$, let C_p be the boundary component of \mathbf{S}_P created in this way. We then introduce a new marked point M_p on each component C_p , and set

$$\mathbf{M}_P = (\mathbf{M} \setminus P) \cup \{M_p\}_{p \in P},$$

creating a new bordered surface with marked points $(\mathbf{S}_P, \mathbf{M}_P)$. The sets of marked points \mathbf{M}_P and \mathbf{M} can be identified with each other in a natural way. See Figure 20.

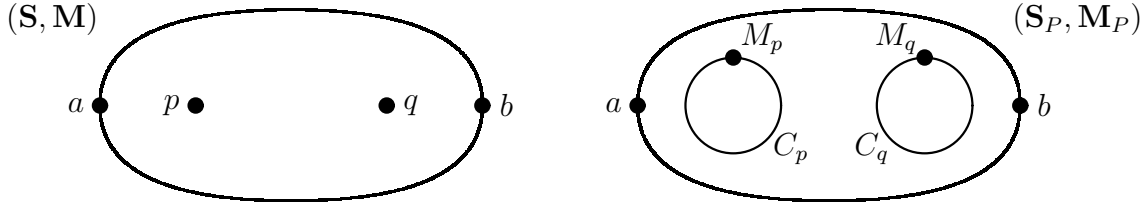


FIGURE 20. Opening of a surface. Here (\mathbf{S}, \mathbf{M}) is a twice-punctured digon, $\mathbf{M} = \{a, b, p, q\}$, $P = \overline{\mathbf{M}} = \{p, q\}$, and $\mathbf{M}_P = \{a, b, M_p, M_q\}$.

Remark 9.2. Be careful to distinguish a surface with an opening from a surface with an extra boundary component with one marked point. We will consider different Teichmüller spaces in the two cases, and treat them rather differently.

There is a natural “projection” map

$$(9.1) \quad \kappa_P : \mathbf{A}^\circ(\mathbf{S}_P, \mathbf{M}_P) \rightarrow \mathbf{A}^\circ(\mathbf{S}, \mathbf{M})$$

(surjective but not injective) that corresponds to collapsing the new boundary components C_p . We will refer to any $\overline{\gamma} \in \mathbf{A}^\circ(\mathbf{S}_P, \mathbf{M}_P)$ that projects onto a given arc

$\gamma \in \mathbf{A}^\circ(\mathbf{S}, \mathbf{M})$ as a *lift* of γ . To describe these lifts, we introduce, for every $p \in P$, the map

$$(9.2) \quad \psi_p : \mathbf{A}^\circ(\mathbf{S}_P, \mathbf{M}_P) \rightarrow \mathbf{A}^\circ(\mathbf{S}_P, \mathbf{M}_P)$$

that takes each arc ending at M_p and twists it once clockwise around C_p (with a negative Dehn twist). Then, for example, an arc $\gamma \in \mathbf{A}^\circ(\mathbf{S}, \mathbf{M})$ connecting two distinct punctures $p, q \in P$ has the lifts

$$(9.3) \quad \kappa_P^{-1}(\gamma) = \{(\psi_p)^n(\psi_q)^m \bar{\gamma}\}_{n,m \in \mathbb{Z}},$$

where $\bar{\gamma}$ is some particular lift of γ . See Figure 21. If $\gamma \in \mathbf{A}^\circ(\mathbf{S}, \mathbf{M})$ goes from a puncture $p \in P$ back to itself, then $\kappa_P^{-1}(\gamma)$ consists of two orbits under the action of ψ_p .

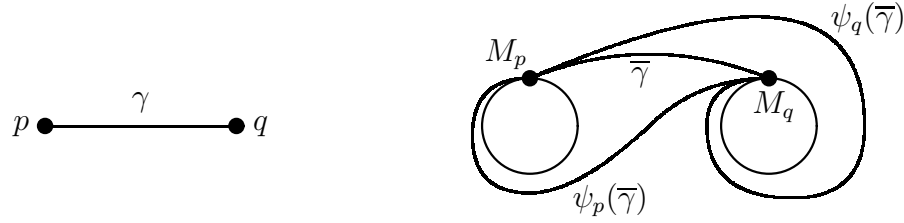


FIGURE 21. An arc connecting two punctures, and three of its lifts

Opening all the punctures in $\overline{\mathbf{M}}$ results in the “largest” opened surface

$$(9.4) \quad (\overline{\mathbf{S}}, \overline{\mathbf{M}}) = (\mathbf{S}_{\overline{\mathbf{M}}}, \mathbf{M}_{\overline{\mathbf{M}}}).$$

Its arc complex

$$(9.5) \quad \mathbf{A}^\circ(\overline{\mathbf{S}}, \overline{\mathbf{M}}) \stackrel{\text{def}}{=} \mathbf{A}^\circ(\mathbf{S}_{\overline{\mathbf{M}}}, \mathbf{M}_{\overline{\mathbf{M}}})$$

naturally projects onto all the other arc complexes $\mathbf{A}^\circ(\mathbf{S}_P, \mathbf{M}_P)$; that is, the map $\kappa_{\overline{\mathbf{M}}}$ factors through every other map κ_P , for $P \subset \overline{\mathbf{M}}$.

Definition 9.3 (*Lifts of tagged arcs*). To lift a tagged arc $\gamma \in \mathbf{A}^\circ(\mathbf{S}, \mathbf{M})$ to an opened surface $(\mathbf{S}_P, \mathbf{M}_P)$ (in particular, to $(\overline{\mathbf{S}}, \overline{\mathbf{M}})$), we simply lift the untagged version of γ to $\mathbf{A}^\circ(\mathbf{S}_P, \mathbf{M}_P)$ (resp., $\mathbf{A}^\circ(\overline{\mathbf{S}}, \overline{\mathbf{M}})$), and then affix the same tags as the ones used at the corresponding ends of γ . See Figure 22. Thus, the lifted tagged arc $\bar{\gamma}$ may have a notched end at an unopened point $p \notin P$, or at a marked point M_p ($p \in P$). We denote by $\mathbf{A}^\circ(\mathbf{S}_P, \mathbf{M}_P)$ (resp., $\mathbf{A}^\circ(\overline{\mathbf{S}}, \overline{\mathbf{M}})$) the set of all such tagged arcs $\bar{\gamma}$ on $(\mathbf{S}_P, \mathbf{M}_P)$ (resp., $(\overline{\mathbf{S}}, \overline{\mathbf{M}})$). Note that as in $\mathbf{A}^\circ(\mathbf{S}, \mathbf{M})$, tagged arcs that enclose a monogon containing a single puncture p are forbidden, whether or not $p \in P$.

10. LAMBDA LENGTHS ON OPENED SURFACES

We are now prepared to describe our main Teichmüller-theoretic construction. This will be done in two steps, Definitions 10.2 and 10.8.

Definition 10.1. A *decorated set of marked points* \tilde{P} is a subset $P \subset \overline{\mathbf{M}}$ of the punctures, together with a choice of orientation on C_p for each $p \in P$; this orientation can be clockwise or counterclockwise.



FIGURE 22. Lift of a tagged arc

Definition 10.2. Fix a decorated set of marked points \tilde{P} . We define the *partially opened Teichmüller space* $\mathcal{T}_{\tilde{P}}(\mathbf{S}_P, \mathbf{M}_P)$ as the space of all finite-volume, complete hyperbolic metrics on $\mathbf{S}_P \setminus (\mathbf{M} \setminus P)$ with geodesic boundary, modulo isotopy. For a decorated set of marked points \tilde{P} , the *decorated partially opened Teichmüller space* $\tilde{\mathcal{T}}_{\tilde{P}}(\mathbf{S}_P, \mathbf{M}_P)$ is the same set of metrics as in $\mathcal{T}_{\tilde{P}}(\mathbf{S}_P, \mathbf{M}_P)$, modulo isotopy relative to $\{M_p\}_{p \in P}$ and with a choice of horocycle around each point in $\mathbf{M} \setminus P$.

(The orientations on the boundary will be used shortly.)

That is, a hyperbolic structure in $\mathcal{T}_{\tilde{P}}(\mathbf{S}_P, \mathbf{M}_P)$ has a cusp at each point in $\mathbf{M} \setminus P$ (i.e., at each original marked point on $\partial \mathbf{S}$ and at each puncture in $\overline{\mathbf{M}} \setminus P$), and a new circular geodesic boundary component C_p arising from each point $p \in P$. The boundary is otherwise geodesic. Note in particular that there are no cusps at the points M_p , which are introduced merely to help parametrize the new boundary.

Remark 10.3. The decorated space $\tilde{\mathcal{T}}_{\tilde{P}}(\mathbf{S}_P, \mathbf{M}_P)$ is a fibration over $\mathcal{T}_{\tilde{P}}(\mathbf{S}_P, \mathbf{M}_P)$ with fibers $\mathbb{R}^{\mathbf{M}}$. The decorations look different depending on the marked point: for points not in P , the decoration is a choice of horocycle as in the previous sections, while at the new geodesic boundary the decoration comes from restricting the isotopies to those that leave the boundary components C_p ($p \in P$) fixed. The difference (isotopies of \mathbf{S}_P that are isotopic to the identity, but not while fixing the C_p) is isotopies that twist the surface around the C_p . Such isotopies have a single real parameter for each $p \in P$, namely the amount of twisting.

Given a decorated set of marked points \tilde{P} and a geometric structure $\sigma \in \mathcal{T}_{\tilde{P}}(\mathbf{S}_P, \mathbf{M}_P)$ to each arc $\gamma \in \mathbf{A}^\circ(\mathbf{S}, \mathbf{M})$, we can associate a unique infinite, non-selfintersecting geodesic γ_σ on \mathbf{S}_P (geodesic with respect to σ): at endpoints of γ that are not opened, the geodesic γ_σ runs out to the cusp, while at endpoints that are in P , it spirals around C_p in the chosen direction. See Figure 23.

For an arc $\bar{\gamma} \in \mathbf{A}^\circ(\mathbf{S}_P, \mathbf{M}_P)$ on an opened surface, we set $\bar{\gamma}_\sigma = \gamma_\sigma$, where $\gamma = \kappa_P(\bar{\gamma})$ is obtained from $\bar{\gamma}$ by the collapsing map κ_P of (9.1). Thus the geodesic representative $\bar{\gamma}_\sigma$ does not depend on how much $\bar{\gamma}$ winds around its opened ends.

We now coordinatize the Teichmüller spaces $\tilde{\mathcal{T}}_{\tilde{P}}(\mathbf{S}_P, \mathbf{M}_P)$ by introducing appropriate generalizations of Penner's lambda lengths. This will require the following notion.

Definition 10.4. Fix a decorated set of marked points \tilde{P} and a geometric structure $\sigma \in \tilde{\mathcal{T}}_{\tilde{P}}(\mathbf{S}_P, \mathbf{M}_P)$. For each $p \in P$, there is a *perpendicular horocyclic segment* h_p

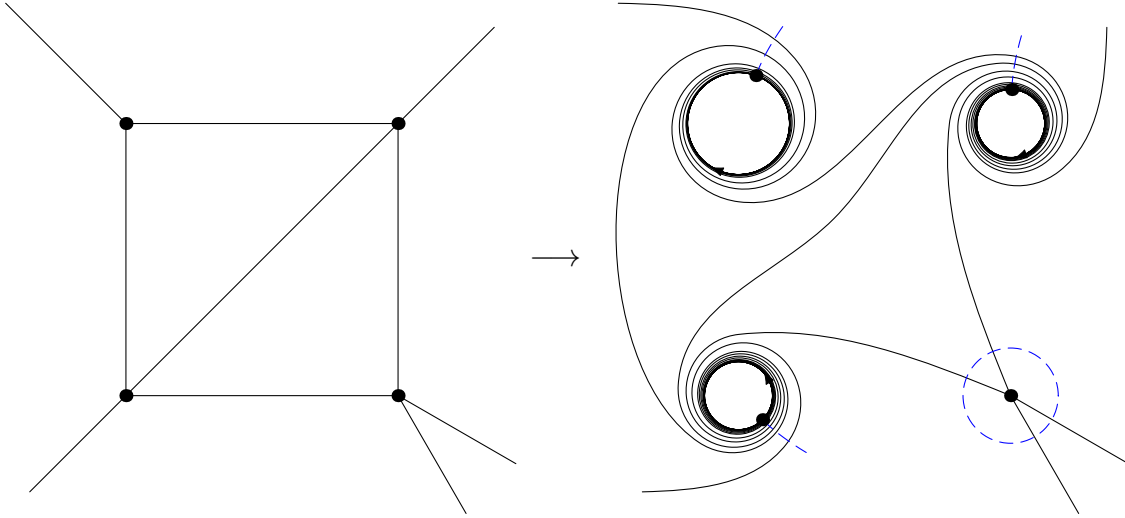


FIGURE 23. Representing arcs by geodesics on an opened surface. Shown on the left is a portion of the original surface; on the right is a particular opening of the surface, endowed with a hyperbolic structure and an orientation on the opened boundary components. The lower right marked point, the only one not in P , has been left as an interior cusp; the remaining three have been opened into circular geodesic boundary components. The lower left component is oriented counterclockwise and the other two are oriented clockwise.

near C_p : we take a (short) segment of the horocycle from $M_p \in C_p$ which is perpendicular to C_p and to all geodesics γ_σ that spiral to C_p in the direction given by the chosen orientation. In Figure 23, the horocycle segments h_p are drawn as dashed curves, as is the horocycle decorating the marked point not in P .

Remark 10.5. The perpendicular horocyclic segment can be obtained by following the *horocyclic flow* from M_p perpendicular to the boundary. (The horocyclic flow is similar to the geodesic flow, but follows the unique horocycle through a given point in a given direction.) It can be thought of as the set of points an equal distance to the ideal point obtained by following the boundary C_p infinitely far in the direction of its orientation. See, e.g., [23] for more on the horocyclic flow.

Definition 10.6 (*Lambda lengths on an opened surface*). We next define lambda lengths $\lambda_\sigma(\gamma)$ for arcs $\gamma \in \mathbf{A}^\circ(\mathbf{S}_P, \mathbf{M}_P)$ and for $\sigma \in \tilde{\mathcal{T}}_P(\mathbf{S}_P, \mathbf{M}_P)$. As in Definition 7.3, we set

$$(10.1) \quad \lambda(\gamma) = \lambda_\sigma(\gamma) = e^{l(\gamma)/2},$$

where $l(\gamma) = l_\sigma(\gamma)$ is the distance between appropriate intersections of the geodesic γ_σ with the horocycles at its two ends. At ends of γ_σ that spiral around one of the openings C_p , there will be many intersections between γ_σ and the horocyclic segment h_p , and we need to pick one of them. Assume that γ connects two ends

M_p and M_q , with both p and q in P (this is the most complicated case). Suppose furthermore that γ twists sufficiently far in the direction of the orientation of the boundary. Then there are unique intersections between γ_σ and each of h_p and h_q such that the path that runs

- along h_p from M_p to one intersection, then
- along γ_σ to the other intersection, then
- along h_q to the other endpoint M_q

is homotopic to the original arc γ , as shown in Figure 24. If one or both of the ends of γ are not in P , we leave γ_σ unmodified at that end and pick the unique intersection between γ_σ and the corresponding horocycle. In either case, $l(\gamma)$ is the (signed) distance along γ_σ between the chosen intersections with the two horocycles.

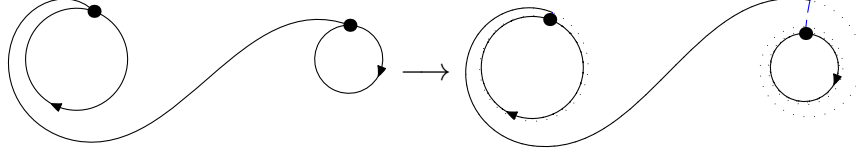


FIGURE 24. Finding the correct intersection with the perpendicular horocycles.

In order to extend the definition to *all* arcs $\gamma \in \mathbf{A}^\circ(\mathbf{S}_P, \mathbf{M}_P)$, not just those that twist sufficiently much, we postulate how $l(\gamma)$ and $\lambda(\gamma)$ change when we twist γ around the boundary. Specifically, we mandate that

$$(10.2) \quad l(\psi_p \gamma) = n_p(\gamma) l(p) + l(\gamma)$$

where

$$(10.3) \quad \psi_p \text{ is the clockwise twist defined by (9.2),}$$

$$(10.4) \quad n_p(\gamma) \text{ is the number of ends of } \gamma \text{ that touch } M_p, \text{ and}$$

$$(10.5) \quad l(p) = \begin{cases} -\text{length of } C_p & \text{if } p \in P \text{ and } C_p \text{ is oriented counterclockwise;} \\ 0 & \text{if } p \notin P; \\ \text{length of } C_p & \text{if } p \in P \text{ and } C_p \text{ is oriented clockwise.} \end{cases}$$

Accordingly (cf. (10.1)), we have

$$(10.6) \quad \lambda(\psi_p \gamma) = \lambda(p)^{n_p(\gamma)} \lambda(\gamma),$$

where

$$(10.7) \quad \lambda(p) = \lambda_\sigma(p) = e^{l(p)/2}.$$

In order for $l(\gamma)$ and $\lambda(\gamma)$ to be well defined, we need of course to check that the requirements (10.2)–(10.6) are consistent with the earlier definitions given in the case where γ twists sufficiently much. This follows from the following lemma.

Lemma 10.7. *The distance along a geodesic γ_σ between successive intersections with the horocycle h_p is always equal to $|l(p)|$.*

Proof. Consider a segment s of γ_σ between successive intersections with h_p which is very close to C_p . Because of the spiraling nature of γ_σ , as the distance from s to C_p approaches 0, the length of s approaches the length of C_p . But since γ_σ is part of a family of geodesics perpendicular to h_p , we can move s within the family of geodesics without changing the length. Now move s out from C_p until it coincides with the desired segment.

Alternatively, a purely geometric proof is sketched in Figure 25. Consider the universal cover \mathbb{H}^2 of (\mathbf{S}, \mathbf{M}) and one lift \tilde{C}_p of C_p within it. Place the endpoint of \tilde{C}_p to which γ_σ is spiraling at infinity in the upper-half-space model of \mathbb{H}^2 . Then the lifts \tilde{h}_p of h_p appear as straight lines parallel to the real axis, and a lift $\tilde{\gamma}_\sigma$ of γ_σ appears as a line parallel to the imaginary axis (and \tilde{C}_p). The distance between successive intersections on $\tilde{\gamma}_\sigma$ is independent of the left-right position of $\tilde{\gamma}_\sigma$, and in particular it agrees with the distance along \tilde{C}_p , namely $|l(p)|$. \square

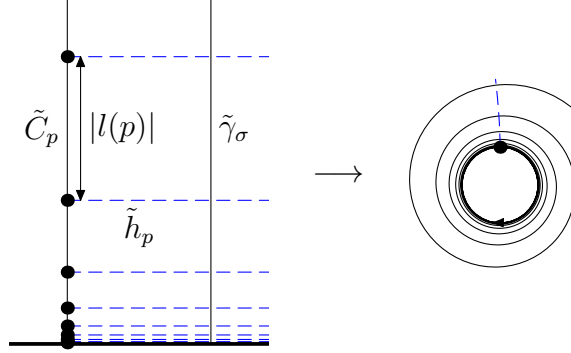


FIGURE 25. Shown on the right is a geodesic γ_σ spiraling to a boundary component C_p , and the corresponding horocycle h_p . On the left, their respective universal covers $\tilde{\gamma}_\sigma$, \tilde{C}_p , and \tilde{h}_p , in the upper-half-space model of \mathbb{H}^2 . The distance between horocycles is the length of C_p .

Definition 10.8. The *complete decorated Teichmüller space* $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M})$ is the disjoint union over all $3^{|\overline{\mathbf{M}}|}$ possible decorated sets of marked points \tilde{P} of $\tilde{\mathcal{T}}_{\tilde{P}}(\mathbf{S}_P, \mathbf{M}_P)$.

It remains to describe the topology on $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M})$. For an arc $\gamma \in \mathbf{A}^\circ(\overline{\mathbf{S}}, \overline{\mathbf{M}})$ (see Definition 9.3), define the lambda length

$$\lambda(\gamma) : \overline{\mathcal{T}}(\mathbf{S}, \mathbf{M}) \rightarrow \mathbb{R}$$

on each stratum of $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M})$ by projecting γ to the appropriate set $\mathbf{A}^\circ(\mathbf{S}_P, \mathbf{M}_P)$ and using the construction above. The topology on $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M})$ (making it into a connected space) is the weakest in which $\lambda(\overline{\gamma})$ is continuous for all lifted arcs $\overline{\gamma} \in \mathbf{A}^\circ(\overline{\mathbf{S}}, \overline{\mathbf{M}})$.

Lemma 10.9. *Inside a quadrilateral in $(\overline{\mathbf{S}}, \overline{\mathbf{M}})$ with sides $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$, and $\overline{\delta}$ and diagonals $\overline{\eta}$ and $\overline{\theta}$ as in Proposition 7.6, we have the Ptolemy relation*

$$(10.8) \quad \lambda(\overline{\eta})\lambda(\overline{\theta}) = \lambda(\overline{\alpha})\lambda(\overline{\gamma}) + \lambda(\overline{\beta})\lambda(\overline{\delta}).$$

Note that the arcs $\bar{\alpha}, \dots, \bar{\theta}$ have to form a quadrilateral in $(\overline{\mathbf{S}}, \overline{\mathbf{M}})$; it is not enough for their projections to (\mathbf{S}, \mathbf{M}) to form a quadrilateral. In particular, if an arc appears twice on the boundary of a quadrilateral in (\mathbf{S}, \mathbf{M}) , we may have to take two different lifts of it to $(\overline{\mathbf{S}}, \overline{\mathbf{M}})$ in order for Lemma 10.9 to apply.

Proof. This is equivalent to Proposition 7.6: the geometry is identical to what we had before, once a lift to the universal cover is made. \square

Proposition 10.10. *Let T be an ideal triangulation of (\mathbf{S}, \mathbf{M}) without self-folded triangles. For each $\gamma \in T$, fix an arc $\bar{\gamma} \in \mathbf{A}^\circ(\overline{\mathbf{S}}, \overline{\mathbf{M}})$ that projects to γ . Then the map*

$$\Phi = \left(\prod_{p \in \overline{\mathbf{M}}} \lambda(p) \right) \times \left(\prod_{\beta \in \mathbf{B}(\mathbf{S}, \mathbf{M})} \lambda(\beta) \right) \times \left(\prod_{\gamma \in T} \lambda(\bar{\gamma}) \right) : \overline{\mathcal{T}}(\mathbf{S}, \mathbf{M}) \rightarrow \mathbb{R}_{>0}^{n+|\mathbf{M}|}$$

is a homeomorphism, where n is the number of arcs in T as in formula (5.1).

Proof. For any vector Λ in $\mathbb{R}_{>0}^{n+|\mathbf{M}|}$, we can construct $\Phi^{-1}(\Lambda)$, the unique geometric structure in $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M})$ with the corresponding set of lambda lengths, as follows. First note that if the arc γ has distinct endpoints p and q in $\overline{\mathbf{M}}$, then for any alternate lift $\bar{\gamma}'$ of γ there are $n, m \in \mathbb{Z}$ so that

$$\bar{\gamma}' = \psi_p^n \psi_q^m(\bar{\gamma})$$

(cf. (9.3)). Then by equation (10.6), for any hyperbolic structure with these lambda coordinates, we have

$$(10.9) \quad \lambda(\bar{\gamma}') = \lambda(p)^n \lambda(q)^m \lambda(\bar{\gamma}).$$

This equation and similar ones can then be used to compute lambda lengths of all lifts of the arcs in T .

Now for each ideal triangle in T with sides $\gamma_1, \gamma_2, \gamma_3$, pick lifts $\bar{\gamma}'_1, \bar{\gamma}'_2, \bar{\gamma}'_3$ that form a triangle in $(\overline{\mathbf{S}}, \overline{\mathbf{M}})$. Then take a decorated ideal hyperbolic triangle (i.e., a triangle with choice of horocycles around each cusp) so that the lambda lengths of the sides of the triangle match with the $\lambda(\bar{\gamma}'_i)$ (computed using (10.9)), as in Penner's proof of Theorem 7.4. (There is a unique decorated ideal hyperbolic triangle with given lambda lengths.)

We next need to glue these triangles together to form a hyperbolic surface. For each arc $\gamma \in T$, we have two different lifts $\bar{\gamma}', \bar{\gamma}''$ of γ coming from the two different triangles that have this arc as a side (or the two different sides of the same triangle, in case γ is the repeated edge of a self-folded triangle). Suppose that $\bar{\gamma}'' = \psi_p^n \psi_q^m(\bar{\gamma}')$, where p and q are the endpoints of γ as before. Then glue the two hyperbolic triangles so that the horocycles around the vertex corresponding to p are offset by $n \cdot l(p)$ and the horocycles around the vertex q are offset by $m \cdot l(q)$. It is then elementary to verify that the resulting glued surface has a metric completion which is a surface with the desired lambda lengths, proving the surjectivity of Φ . Conversely, since each decorated ideal triangle is determined by its lambda lengths and the gluings between adjacent triangles are determined by the data, Φ is injective.

By definition of the topology on $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M})$, the map Φ is continuous. It remains to show that Φ^{-1} is continuous. To do this, we must show that for an arbitrary

arc $\bar{\alpha} \in \mathbf{A}^\circ(\overline{\mathbf{S}}, \overline{\mathbf{M}})$, the lambda length $\lambda(\bar{\alpha})$ is a continuous function of the given coordinates. Let $\alpha = \varkappa(\bar{\alpha})$ be the arc in $\mathbf{A}^\circ(\mathbf{S}, \mathbf{M})$ corresponding to $\bar{\alpha}$. We can move from T to a triangulation that contains α by a series of edge flips in quadrilaterals. For each such flip, Lemma 10.9 lets us write the lambda length of one lift of the new diagonal in terms of lambda lengths of lifts of the old arcs. (If we can write one lift of a given arc in terms of the given lambda coordinates, we can write all lifts in terms of these lambda coordinates by multiplying by appropriate powers of the $\lambda(p)$ for $p \in \overline{\mathbf{M}}$.) We end up inductively writing $\lambda(\bar{\alpha})$ as an algebraic function with non-zero denominator in terms of the original coordinates. Thus each $\lambda(\bar{\alpha})$ is a continuous function when pulled back to $\mathbb{R}^{n+|\mathbf{M}|}$, so by definition of the topology on $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M})$ it follows that Φ^{-1} is continuous. \square

We next wish to extend Proposition 10.10 to the case of tagged triangulations, as in Remark 8.8.

Definition 10.11 (*Lambda lengths of tagged arcs on an opened surface*). For $\sigma \in \overline{\mathcal{T}}(\mathbf{S}, \mathbf{M})$ and $\gamma \in \mathbf{A}^\infty(\mathbf{S}, \mathbf{M})$, define γ_σ to be the unique infinite, non-selfintersecting geodesic which at each notched end spirals *against* the orientation chosen on C_p and is otherwise as before. For $p \in \overline{\mathbf{M}}$, set

$$(10.10) \quad \nu(p) = 2 \ln |\lambda(p) - \lambda(p)^{-1}|.$$

Let \overline{M}_p be the point on C_p a (signed) distance of $\nu(p)$ from M_p in the direction against the orientation of C_p . Define the *conjugate perpendicular horocycle* \overline{h}_p to be the horocycle passing through \overline{M}_p and perpendicular to C_p and to all geodesics spiraling against the orientation on C_p . Finally, for $\overline{\gamma} \in \mathbf{A}^\infty(\overline{\mathbf{S}}, \overline{\mathbf{M}})$, define $l(\overline{\gamma})$ to be the length between intersections with horocycles as before, using the conjugate perpendicular horocycle for notched ends that meet an opened puncture. Specifically, if $\overline{\gamma}$ is notched at p and plain at q , there is a unique path that is homotopic to $\overline{\gamma}$ and runs

- along C_p from M_p to \overline{M}_p a distance of $\nu(p)$ against the orientation of C_p , then
- along \overline{h}_p from \overline{M}_p to an intersection with $\overline{\gamma}_\sigma$, then
- along $\overline{\gamma}_\sigma$ to an intersection with h_q , then
- along h_q to M_q .

There is a similar path if $\overline{\gamma}$ is notched at both ends. Set $\lambda(\overline{\gamma}) = e^{l(\overline{\gamma})/2}$ as before.

We will also allow the obvious extensions of $l(\overline{\gamma})$ and $\lambda(\overline{\gamma})$ to a version of tagged arcs on $(\overline{\mathbf{S}}, \overline{\mathbf{M}})$ which enclose punctured monogons (so are not in $\mathbf{A}^\infty(\overline{\mathbf{S}}, \overline{\mathbf{M}})$).

Remark 10.12. The correction term $\nu(p)$ is chosen so that Lemma 10.14 below comes out with no correction factors, which in turn implies that for each lifted arc $\overline{\gamma} \in \mathbf{A}^\infty(\overline{\mathbf{S}}, \overline{\mathbf{M}})$, the lambda length $\lambda(\overline{\gamma})$ is a continuous function on $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M})$.

As $l(p)$ approaches $\pm\infty$, $\nu(p)$ is asymptotic to $|l(p)|$, which amounts to saying that in the limit as $l(p)$ gets large, \overline{M}_p differs from M_p by a full turn against the orientation on C_p . On the other hand, for $l(p)$ close to zero (when the boundary is close to a cusp), $\nu(p)$ is asymptotic to $2 \ln |l(p)|$, which is large and negative.

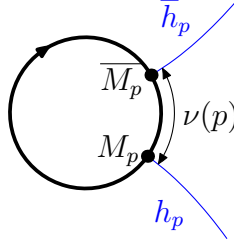


FIGURE 26. The effect of $\nu(p)$ on the horocycles. Shown are the horocycle h_p and the conjugate horocycle \overline{h}_p in the case when $l(p) > 0$ (so C_p is oriented clockwise), and $l(p)$ is large enough so that $\nu(p) > 0$. (Here $l(p) \approx 1.05$ so $\nu(p) \approx 0.2$.)

Question 10.13. Is there a more geometrically natural way to define the conjugate perpendicular horocycle (Definition 10.11)?

We next investigate the properties of the lambda lengths of tagged arcs. In order to complete our construction of exchange patterns associated with opened surfaces, we need to define exchange relations involving

- (i) the lambda lengths $\lambda(\overline{\gamma})$, for $\overline{\gamma} \in \mathbf{A}^\bowtie(\overline{\mathbf{S}}, \overline{\mathbf{M}})$,
- (ii) the lambda lengths $\lambda(\beta)$, for $\beta \in \mathbf{B}(\mathbf{S}, \mathbf{M})$, and
- (iii) the lambda lengths $\lambda(p)$, for $p \in \overline{\mathbf{M}}$.

Some of these relations are easy to obtain. We see right away that the lambda lengths of types (i) and (ii) still obey the Ptolemy relation (Lemma 10.9), provided the arcs form a quadrilateral on $(\overline{\mathbf{S}}, \overline{\mathbf{M}})$ as before and the tags of the three arcs meeting at each vertex of the quadrilateral agree with each other.

Next, there is change of the lift. Equation (10.6) holds as before, with the convention that $n_p(\gamma)$ (cf. (10.4)) is a *signed* count: a plain end of γ at p contributes $+1$, a notched end contributes -1 :

$$(10.11) \quad n_p(\gamma) \text{ is the signed number of ends of } \gamma \text{ that touch } M_p$$

$$(10.12) \quad \lambda(\psi_p \gamma) = \lambda(p)^{n_p(\gamma)} \lambda(\gamma)$$

We will also need a relation associated with a tagged flip inside an opened monogon, an analogue of Lemma 7.10.

Lemma 10.14. *Inside an opened surface, consider a monogon with a marked vertex q and a single boundary component C_p in the interior. Let δ and ϱ be two compatible parallel tagged arcs in $\mathbf{A}^\bowtie(\overline{\mathbf{S}}, \overline{\mathbf{M}})$ connecting q and M_p , with δ plain and ϱ notched at M_p , as shown in Figure 27 on the left, and let η be the outer boundary of the monogon, tagged like δ and ϱ at q . Then*

$$(10.13) \quad \lambda(\delta)\lambda(\varrho) = \lambda(\eta).$$

Proof. Let \tilde{h}_p be the horocycle which is like \overline{h}_p but perpendicular to C_p at M_p instead of \overline{M}_p as in Figure 28, and let θ be the tagged arc like ϱ but with lambda length

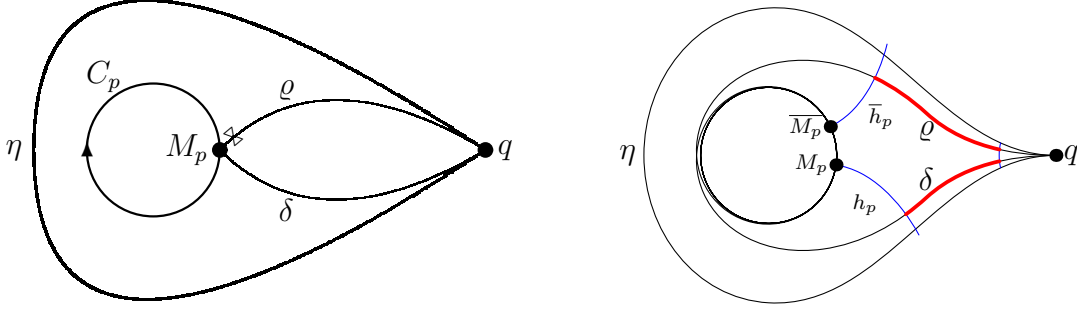


FIGURE 27. An opened monogon

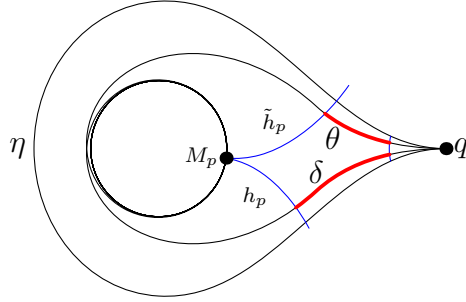


FIGURE 28. The opened monogon with alternate horocycles

measured with respect to \tilde{h}_p . Then (10.13) is equivalent to

$$(10.14) \quad \lambda(\delta)\lambda(\theta) = \frac{\lambda(\eta)}{|\lambda(p) - \lambda(p)^{-1}|}.$$

Strange as it may seem at first glance, (10.14) is yet another instance of the same Ptolemy relation. To see that, suppose first that C_p is oriented clockwise (so that $\lambda(p) > 1$, cf. (10.5), (10.7)), and consider Figure 29, which on the top shows lifts of the arcs δ , θ , and η to the universal cover of the monogon. The bottom of Figure 29 shows a different triple of lifts together with lifts of the arcs $\delta' = \psi_p \delta$ and $\theta' = \psi_p^{-1} \theta$. By applying the Ptolemy relation to the quadrilateral with diagonals $\bar{\delta}'$ and $\bar{\theta}'$, we obtain

$$\lambda(\delta')\lambda(\theta') = \lambda(\delta)\lambda(\theta) + \lambda(p)\lambda(\eta).$$

Combining this with

$$\lambda(\delta') = \lambda(p) \lambda(\delta)$$

$$\lambda(\theta') = \lambda(p) \lambda(\theta)$$

(from Equation (10.12)) we deduce

$$(\lambda(p)^2 - 1)\lambda(\delta)\lambda(\theta) = \lambda(p) \lambda(\eta)$$

as desired.

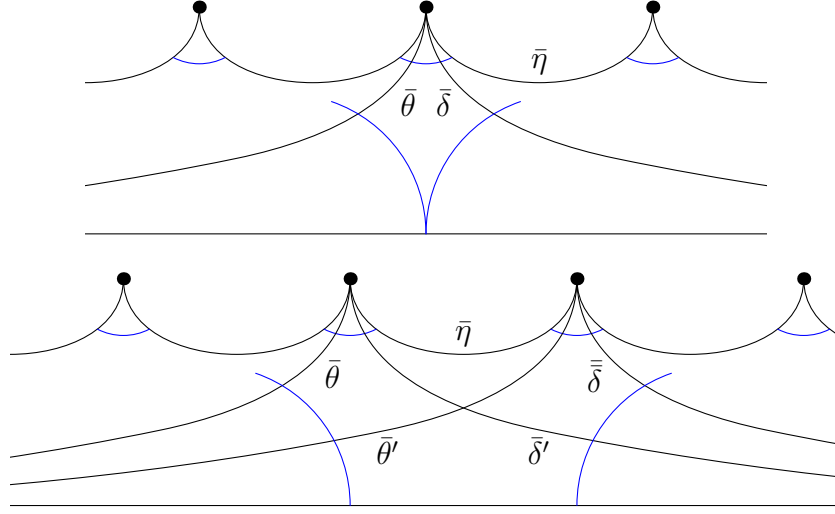


FIGURE 29. The universal cover of the opened monogon, with different choices of lifts.

If C_p is oriented counterclockwise instead, δ spirals counterclockwise and ϱ and θ spiral clockwise. In this case define $\delta' = \psi_p^{-1}\delta$ and $\theta' = \psi_p\theta$. Then we have

$$\lambda(\delta')\lambda(\theta') = \lambda(\delta)\lambda(\theta) + \lambda(\eta)\lambda(p)^{-1}$$

and we again deduce (10.14). \square

Lemma 10.14 lets us find the relations associated with the tagged flips of type (5.D).

Lemma 10.15. *Consider an opened digon with vertices r and q and an opening C_p with a marked point M_p . Let α , β , ϱ , and θ be the tagged arcs shown in Figure 30. (Possible tags at r and q have been suppressed in the picture.) We assume that the arcs in $\{\alpha, \beta, \varrho, \theta\}$ are tagged so that any two of them are compatible, with the exception of the pair (ϱ, θ) . Then*

$$(10.15) \quad \lambda(\varrho)\lambda(\theta) = \lambda(\alpha) + \lambda(p)^{-1}\lambda(\beta).$$

Proof. Let us introduce the arcs γ , δ , and η as in Figure 31 (all tagged plain at p). The lambda lengths of the six arcs in Figure 31 satisfy the Ptolemy relation (7.2):

$$\lambda(\theta)\lambda(\eta) = \lambda(\alpha)\lambda(\gamma) + \lambda(\beta)\lambda(\delta).$$

We also have

$$(10.16) \quad \lambda(\delta) = \lambda(\gamma)\lambda(p)^{-1}$$

(by (10.12)) and

$$(10.17) \quad \lambda(\eta) = \lambda(\gamma)\lambda(\varrho)$$

(by Lemma 10.14). Putting everything together, we obtain (10.15). \square

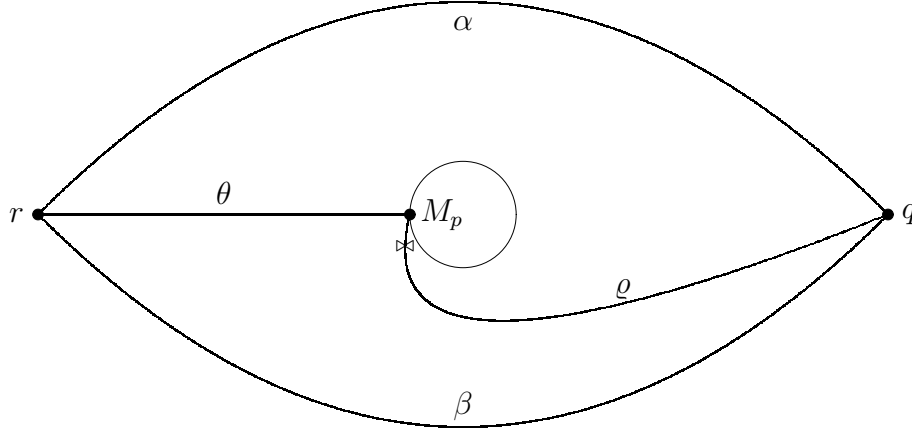


FIGURE 30. An exchange relation in an opened digon.

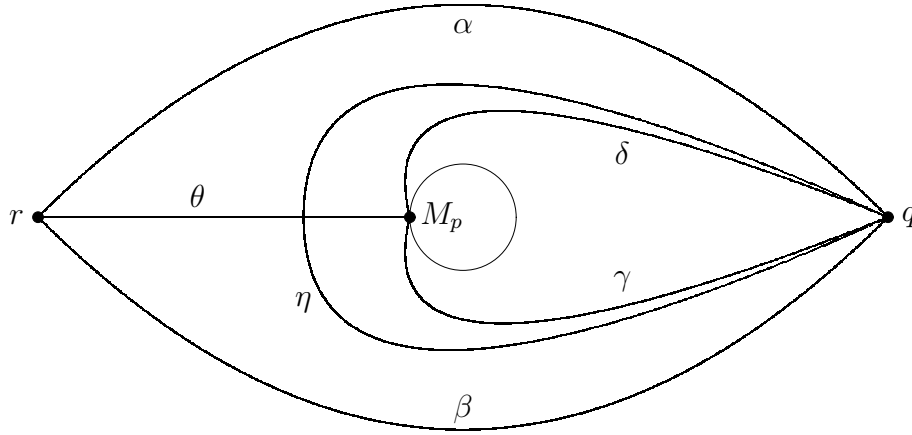


FIGURE 31. Proof of Lemma 10.15.

Corollary 10.16. *Let T be a tagged triangulation of (\mathbf{S}, \mathbf{M}) . For each $\gamma \in T$, fix an arc $\bar{\gamma} \in \mathbf{A}^\infty(\overline{\mathbf{S}}, \overline{\mathbf{M}})$ that projects to γ . Then the map Φ as defined in Proposition 10.10 is a homeomorphism.*

Proof. The triangulation T can be connected to an ideal triangulation T' of (\mathbf{S}, \mathbf{M}) with no notched arcs or self-folded triangles by a sequence of flips in quadrilaterals and digons. At each step, Lemmas 10.9 and 10.14 let us express the lambda lengths after the flip in terms of those before the flip, so the map Φ above and the analogue defined with respect to T' are related by a homeomorphism on the target. But the latter map is a homeomorphism by Proposition 10.10. \square

Remark 10.17. We note that while the definitions of lambda lengths on the opened surface depend in an essential way on the chosen orientations of the boundaries C_p , the relations (10.8), (10.13), and (10.15) that they satisfy have the same form irrespective of the choices of orientations.

11. NON-NORMALIZED EXCHANGE PATTERNS FROM SURFACES

In this section, we describe a construction of a non-normalized exchange pattern on the exchange graph $\mathbf{E}(\mathbf{S}, \mathbf{M})$ of tagged triangulations of the original surface (\mathbf{S}, \mathbf{M}) . This construction is *different* from the one given in Section 8: although it is more complicated, it is eventually going to provide—after proper rescaling—a more general class of coefficients. Here is the basic idea: rather than designating the lambda length of a tagged arc $\gamma \in \mathbf{A}^\infty(\mathbf{S}, \mathbf{M})$ as the corresponding cluster variable, we take the lambda length of an arbitrary lift of γ to the opened surface $(\overline{\mathbf{S}}, \overline{\mathbf{M}})$ (see (9.4) and Figure 21). It turns out that we do not have to coordinate these lifts: the corresponding lambda lengths will always form an exchange pattern. In contrast to the simpler construction in Section 8, this new exchange pattern will not be normalized.

We begin by setting up the coefficient group $\mathbb{P} = \mathbb{P}(\mathbf{S}, \mathbf{M})$ as the (free) abelian multiplicative group generated by the set

$$(11.1) \quad \{\lambda(p) : p \in \overline{\mathbf{M}}\} \cup \{\lambda(\beta) : \beta \in \mathbf{B}(\mathbf{S}, \mathbf{M})\}$$

of lambda lengths of boundary components β and opened circular components C_p . By Proposition 10.10, we can view (and treat) these lambda lengths either as functions on the complete decorated Teichmüller space $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M})$ or as formal variables (=coordinate functions).

For each tagged arc $\gamma \in \mathbf{A}^\infty(\mathbf{S}, \mathbf{M})$, let us fix an arbitrary lift $\overline{\gamma} \in \mathbf{A}^\infty(\overline{\mathbf{S}}, \overline{\mathbf{M}})$ (see Definition 9.3), and set $x(\gamma) = \lambda(\overline{\gamma})$. Then, for each tagged triangulation $T \in \mathbf{E}(\mathbf{S}, \mathbf{M})$, define

$$(11.2) \quad \mathbf{x}(T) = \{x(\gamma) : \gamma \in T\}.$$

In view of Corollary 10.16, the rescaled lambda lengths in $\mathbf{x}(T)$ can be treated as formal variables algebraically independent over the field of fractions of $\mathbb{P}(\mathbf{S}, \mathbf{M})$.

We are now ready to state our next theorem: the lambda lengths of lifts of tagged arcs form a non-normalized exchange pattern.

Theorem 11.1. *For an arbitrary choice of lifts $\overline{\gamma}$ of the tagged arcs $\gamma \in \mathbf{A}^\infty(\mathbf{S}, \mathbf{M})$, there exists a (unique) non-normalized exchange pattern on $\mathbf{E}(\mathbf{S}, \mathbf{M})$ with the following properties:*

- the coefficient group is $\mathbb{P} = \mathbb{P}(\mathbf{S}, \mathbf{M})$;
- the cluster variables are the lambda lengths $\lambda(\overline{\gamma})$;
- the cluster $\mathbf{x}(T)$ at a vertex $T \in \mathbf{E}(\mathbf{S}, \mathbf{M})$ is given by (11.2);
- the ambient field is generated over \mathbb{P} by some (equivalently, any) cluster $\mathbf{x}(T)$;
- the exchange matrices are the signed adjacency matrices $B(T)$; and
- the exchange relations out of each seed are the relations (10.8) and (10.15) associated with the corresponding tagged flips, properly rescaled using (10.12) to reflect the choices of lifts.

To be more accurate, the description of cluster variables above should refer to a “positive realization” of the exchange pattern in question, in the spirit of Definition 4.4. Even though this pattern is not of geometric type (as it is not normalized),

the corresponding notions still have clear meaning, and the analogue of Proposition 4.5 holds.

Proof. The proof is similar to the proof of Theorem 8.6. As before, the real issue is coefficients: we need to demonstrate that they satisfy the requisite mutation rules (2.2)–(2.3). It is straightforward to check that these rules hold for each triple of flips/mutations

$$T_1 \xleftrightarrow{\mu_x} T_2 \xleftrightarrow{\mu_z} T_3 \xleftrightarrow{\mu_x} T_4,$$

if the lifts of the arcs involved are chosen in a coordinated way (this is essentially the same verification as before)—and therefore this rule would hold for any lifts, by Proposition 3.1. \square

12. LAMINATIONS AND SHEAR COORDINATES

In this section, we briefly review a small fragment—as this is all we need—of W. Thurston’s theory of measured laminations [4, 31], and its relationship with matrix mutations. Our exposition is an abridged adaptation of the one given by V. Fock and A. Goncharov [7, Section 3]. An interested reader is referred to the cited sources for further details.

In the next section, we will extend these constructions to the tagged setting.

Definition 12.1. An *integral unbounded measured lamination*—in this paper, frequently just a *lamination*—on a marked surface (\mathbf{S}, \mathbf{M}) is a finite collection of non-selfintersecting and pairwise non-intersecting curves in \mathbf{S} , modulo isotopy relative to \mathbf{M} , subject to the restrictions specified below. Each curve must be one of the following:

- a closed curve (an embedded circle);
- a curve connecting two unmarked points on the boundary of \mathbf{S} ;
- a curve starting at an unmarked point on the boundary and, at its other end, spiraling into a puncture (either clockwise or counterclockwise); or
- a curve both of whose ends spiral into punctures (not necessarily distinct).

Also, the following types of curves are not allowed:

- a curve that bounds an unpunctured or once-punctured disk;
- a curve with two endpoints on the boundary of \mathbf{S} which is isotopic to a piece of boundary containing no marked points, or a single marked point; and
- a curve with two ends spiraling into the same puncture in the same direction without enclosing anything else.

(Note that a curve with two ends spiraling into the same puncture, with the two ends spiraling in opposite directions, is excluded since it is selfintersecting.) See Figures 32 and 33.

Laminations on a marked surface (\mathbf{S}, \mathbf{M}) can be coordinatized using W. Thurston’s shear coordinates.

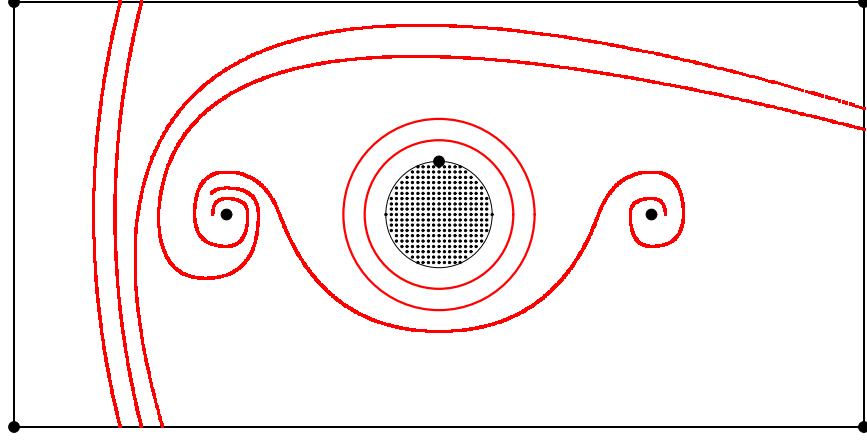


FIGURE 32. A lamination in a twice-punctured annulus with a total of 7 marked points.

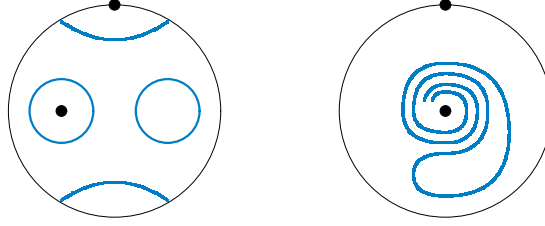


FIGURE 33. Curves that are not allowed in a lamination.

Definition 12.2 (*Shear coordinates*). Let L be an integral unbounded measured lamination. Let T be a triangulation without self-folded triangles. For each arc γ in T , the corresponding *shear coordinate* of L with respect to the triangulation T , denoted by $b_\gamma(T, L)$, is defined as a sum of contributions from all intersections of curves in L with the arc γ . Specifically, such an intersection contributes $+1$ (resp., -1) to $b_\gamma(T, L)$ if the corresponding segment of a curve in L cuts through the quadrilateral surrounding γ cutting through edges in the shape of an ‘S’ (resp., in the shape of a ‘Z’), as shown in Figure 34 on the left (resp., on the right). Note that at most one of these two types of intersection can occur. Note also that even though a spiraling curve can intersect an arc infinitely many times, the number of intersections that contribute to the computation of $b_\gamma(T, L)$ is always finite.

An example is shown in Figure 35.

An alternative (more conceptual) definition of shear coordinates can be given using the notion of tropical lambda lengths, cf. (14.12).

Theorem 12.3 (W. Thurston). *For a fixed triangulation T without self-folded triangles, the map*

$$L \mapsto (b_\gamma(T, L))_{\gamma \in T}$$

is a bijection between integral unbounded measured laminations and \mathbb{Z}^n .

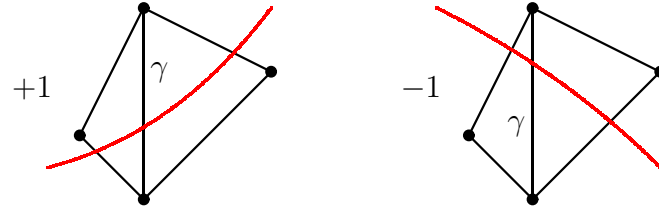


FIGURE 34. Defining the shear coordinate $b_\gamma(T, L)$. The curve on the left contributes $+1$, the one on the right contributes -1 .

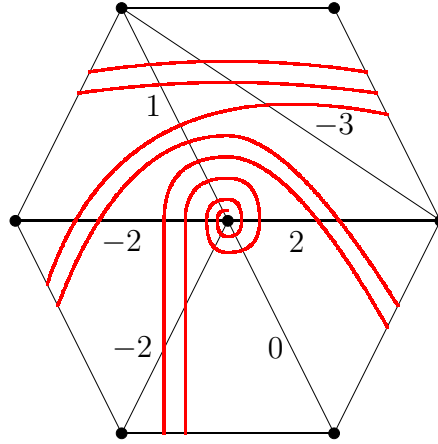


FIGURE 35. Shear coordinates of a lamination L with respect to an ordinary triangulation T .

Example 12.4. Figure 36 shows the six “elementary” laminations L_1, \dots, L_6 of a once-punctured digon (each lamination L_i consisting of a single curve), and their shear coordinates with respect to a particular triangulation T . It is easy to see that any integral lamination of the digon consists of several (possibly none) curves homotopic to some L_i , together with several (possibly none) curves homotopic to $L_{(i+1) \bmod 6}$. It is also easy to see that, in agreement with Theorem 12.3, each vector in \mathbb{Z}^2 can be uniquely written as a non-negative integer linear combination

$$p\mathbf{y}_i + q\mathbf{y}_{(i+1) \bmod 6} \quad (p, q \in \mathbb{Z}_{\geq 0}),$$

where the six vectors

$$\mathbf{y}_1 = [-1, 0], \quad \mathbf{y}_2 = [-1, 1], \quad \mathbf{y}_3 = [0, 1], \quad \mathbf{y}_4 = [1, 0], \quad \mathbf{y}_5 = [1, -1], \quad \mathbf{y}_6 = [0, -1]$$

represent the shear coordinates of L_1, \dots, L_6 , respectively.

Definition 12.5 (*Multi-laminations and associated extended exchange matrices*). A *multi-lamination* is simply a finite family of laminations. Let us fix a multi-lamination

$$\mathbf{L} = (L_{n+1}, \dots, L_m)$$

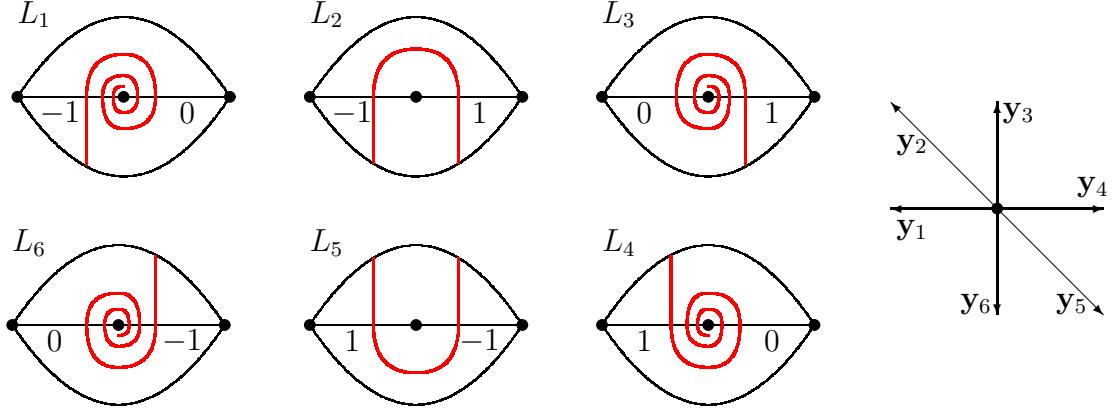


FIGURE 36. Six “elementary” laminations of a once-punctured digon, and the corresponding vectors of shear coordinates.

of size $m - n$; this choice of indexing will be convenient in the sequel. For a triangulation T of (\mathbf{S}, \mathbf{M}) without self-folded triangles, define an $m \times n$ matrix

$$\tilde{B} = \tilde{B}(T, \mathbf{L}) = (b_{ij})$$

(cf. Definition 4.2) as follows. The top $n \times n$ part of \tilde{B} is the signed adjacency matrix $B(T) = (b_{ij})_{1 \leq i, j \leq n}$ (cf. Definition 5.15), whereas the bottom $m - n$ rows are formed by the shear coordinates of the laminations L_i with respect to the triangulation T :

$$(12.1) \quad b_{ij} = b_j(T, L_i) \quad \text{if } n < i \leq m.$$

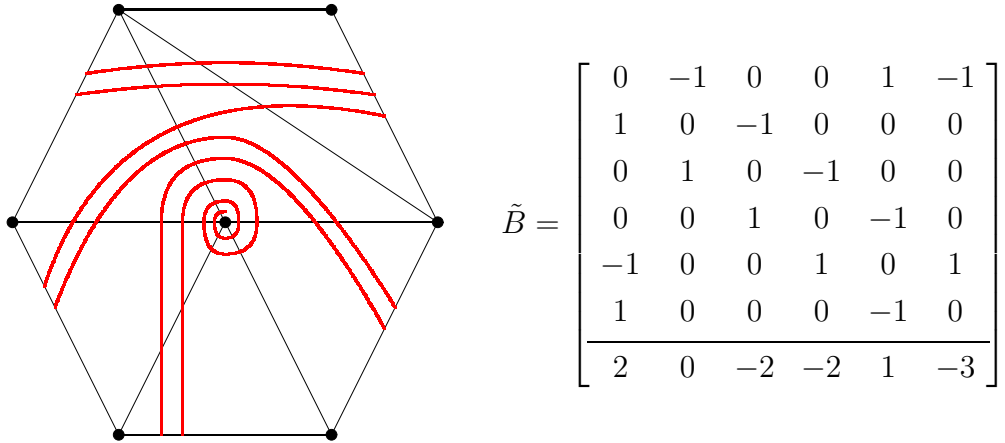


FIGURE 37. The matrix $\tilde{B} = \tilde{B}(T, \mathbf{L})$ for the example in Figure 35. The multi-lamination $\mathbf{L} = (L)$ consists of a single lamination L . We use the labeling of arcs in T shown in Figure 11.

The key observation is that, under ordinary quadrilateral flips, the matrices $\tilde{B}(T)$ transform according to the mutation rules.

Theorem 12.6 (W. Thurston–V. Fock–A. Goncharov). *Fix a multi-lamination \mathbf{L} . If triangulations T and T' without self-folded triangles are related by a flip of an arc k , then the corresponding matrices $\tilde{B}(T, \mathbf{L})$ and $\tilde{B}(T', \mathbf{L})$ are related by a mutation in direction k .*

Although the reader will not find the exact statement of Theorem 12.6 in the work of the above authors, it can be seen to be a version of the results contained therein. More specifically, applying the definition of a matrix mutation to the case under consideration results in identities for the shear coordinates (with respect to T and T') that appear, for example, at the end of [7, Section 3.1]. It is elementary to check these identities directly; in Section 14 we will give a more conceptual proof.

13. SHEAR COORDINATES WITH RESPECT TO TAGGED TRIANGULATIONS

In this section, we define shear coordinates for triangulations with self-folded triangles and, more generally, for tagged triangulations. We then obtain the appropriate analogues of Theorems 12.3 and 12.6.

Definition 13.1 (*Shear coordinates with respect to a tagged triangulation*). We extend Definition 12.2 by defining the *shear coordinates* $b_\gamma(T, L)$ of an integral unbounded measured lamination L with respect to an arbitrary tagged triangulation T . (Here γ runs over the tagged arcs in T .) These coordinates are uniquely defined by the following rules:

- (i) Suppose that tagged triangulations T_1 and T_2 coincide except that at a particular puncture p , the tags of the arcs in T_1 are all different from the tags of their counterparts in T_2 . Suppose that laminations L_1 and L_2 coincide except that each curve in L_1 that spirals into p has been replaced in L_2 by a curve that spirals in the opposite direction. Then $b_{\gamma_1}(T_1, L_1) = b_{\gamma_2}(T_2, L_2)$ for each tagged arc $\gamma_1 \in T_1$ and its counterpart $\gamma_2 \in T_2$.
- (ii) By performing tag-changing transformations $L_1 \rightsquigarrow L_2$ with L_1 and L_2 as above, we can convert any tagged triangulation into a tagged triangulation T that does not contain any notches except possibly inside once-punctured digons. Let T° denote the ideal triangulation that is represented by such T ; that is, $T = \tau(T^\circ)$ in the notation of Definitions 5.8 and 5.10. Let γ° be an arc in T° that is not contained inside a self-folded triangle, and let $\gamma = \tau(\gamma^\circ)$. Then, for a lamination L , we define $b_\gamma(T, L)$ by applying the rule in Definition 12.2 to the ordinary arc γ° viewed inside the triangulation T° .

Note that if γ° is contained inside a self-folded triangle in T° enveloping a puncture p , then we can first apply the tag-changing transformation (i) to T at p , and then use the rule (ii) to determine the shear coordinate in question.

Example 13.2. Let T be the tagged triangulation of a punctured digon shown in Figure 38 (cf. also Figure 36), and let T_1 , T_2 , and T_{12} be the tagged triangulations obtained from T as follows:

- T_1 is obtained from T by the tagged flip replacing γ_{ap} by the tagged arc $\gamma_{b\bar{p}}$;
- T_2 is obtained from T by the tagged flip replacing γ_{bp} by the tagged arc $\gamma_{a\bar{p}}$;

- T_{12} is obtained from T by performing both of these (commuting) tagged flips.

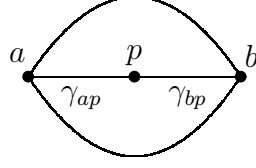


FIGURE 38. A triangulation T of a once-punctured digon.

Let L be a lamination in this once-punctured digon, with shear coordinates $b_1(T, L)$ and $b_2(T, L)$ corresponding to the arcs γ_{ap} and γ_{bp} of T , respectively. We similarly define $b_1(T_s, L)$ and $b_2(T_s, L)$ for each subscript $s \in \{1, 2, 12\}$. For example, $b_1(T_{12}, L)$ and $b_2(T_{12}, L)$ refer to the shear coordinates associated with $\gamma_{b\bar{p}}$ and $\gamma_{a\bar{p}}$, respectively, since these tagged arcs replace γ_{ap} and γ_{bp} , respectively. Then the rules (i)–(ii) of Definition 13.1 yield:

$$b_j(T_s, L) = \begin{cases} -b_j(T, L) & \text{if } j \text{ appears in } s; \\ b_j(T, L) & \text{if } j \text{ does not appear in } s. \end{cases}$$

For example, the shear coordinates of the laminations L_1 and L_3 (in the notation of Figure 36) with respect to the tagged triangulations T, T_1, T_2, T_{12} are:

$$\begin{array}{cccc} b_1(T, L_1) = -1 & b_2(T, L_1) = 0 & b_1(T, L_3) = 0 & b_2(T, L_3) = 1 \\ b_1(T_1, L_1) = 1 & b_2(T_1, L_1) = 0 & b_1(T_1, L_3) = 0 & b_2(T_1, L_3) = 1 \\ b_1(T_2, L_1) = -1 & b_2(T_2, L_1) = 0 & b_1(T_2, L_3) = 0 & b_2(T_2, L_3) = -1 \\ b_1(T_{12}, L_1) = 1 & b_2(T_{12}, L_1) = 0 & b_1(T_{12}, L_3) = 0 & b_2(T_{12}, L_3) = -1 \end{array}$$

We now extend Definition 12.5 to the tagged case.

Definition 13.3. The *extended exchange matrix* $\tilde{B}(T, \mathbf{L})$ of a multi-lamination \mathbf{L} with respect to a tagged triangulation T is defined in exactly the same way as in Definition 12.5, this time using the shear coordinates from Definition 13.1.

Example 13.4. Continuing with Example 13.2, let $\mathbf{L} = (L_1)$. Then, e.g.,

$$\tilde{B}(T, \mathbf{L}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad \tilde{B}(T_1, \mathbf{L}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We are now prepared to provide the promised generalizations of Theorems 12.3 and 12.6.

Theorem 13.5. *Fix a multi-lamination \mathbf{L} . If tagged triangulations T and T' are related by a flip of a tagged arc k , then the corresponding matrices $\tilde{B}(T, \mathbf{L})$ and $\tilde{B}(T', \mathbf{L})$ are related by a mutation in direction k .*

Proof. The proof is a straightforward albeit tedious case-by-case verification based on Theorem 12.6, Definition 13.1, and Remark 5.13. For a flip inside a punctured digon, the analysis involves the laminations L_1, \dots, L_6 from Example 12.4. (The

more delicate part concerns the transformation of the shear coordinates of the arcs on the boundary of the digon.) \square

In Section 14 we will give an alternative (more conceptual and more detailed) proof of Theorem 13.5.

Theorem 13.6. *For a fixed tagged triangulation T , the map*

$$L \mapsto (b_\gamma(T, L))_{\gamma \in T}$$

is a bijection between integral unbounded measured laminations and \mathbb{Z}^n .

Proof. This statement follows from Theorem 12.3, Theorem 13.5, and the invertibility of matrix mutations. Proceed by induction on the number of flips required to obtain T from a triangulation without self-folded triangles. \square

14. TROPICAL LAMBDA LENGTHS

A naïve definition of a tropical lambda length of an arc γ with respect to a (multi-)lamination L is based on the notion of a transverse measure of γ with respect to L , i.e., the corresponding intersection number. The latter notion is however ill defined, as a curve in L spiraling into a puncture p intersects the arcs incident to p infinitely many times. (In the absence of punctures, this issue does not come up.) We bypass this problem by passing to the opened surface where γ is replaced by a family of lifts $\bar{\gamma}$, as we did in Section 10 for lambda lengths. This sets the stage for Section 15, where the tropical lambda lengths of those lifts are used as rescaling factors allowing us to construct the requisite normalized patterns with arbitrary coefficients of geometric type.

Remark 14.1. One can alternatively define tropical lambda lengths via a limiting procedure that degenerates a hyperbolic structure on (\mathbf{S}, \mathbf{M}) into a discrete, or tropical, version thereof, in the spirit of W. Thurston's approach to compactifying Teichmüller spaces. Further hints are provided in Appendix A. In this paper, we do not systematically pursue this course of action, as the limiting objects, non-integral measured laminations, are more complicated than we need.

Definition 14.2. An (integral) *lifted measured lamination* \bar{L} on $(\bar{\mathbf{S}}, \bar{\mathbf{M}}) = (\mathbf{S}_{\bar{\mathbf{M}}}, \mathbf{M}_{\bar{\mathbf{M}}})$ (cf. 9.4) consists of a choice of orientation on each opened puncture C_p together with a finite collection of non-intersecting curves on $(\bar{\mathbf{S}}, \bar{\mathbf{M}})$, modulo isotopy relative to $\mathbf{M}_{\bar{\mathbf{M}}}$, with the following restrictions. First, each component is

- a closed curve, or else
- a curve connecting two points on the boundary of $\mathbf{S}_{\bar{\mathbf{M}}}$ away from $\mathbf{M}_{\bar{\mathbf{M}}}$.

Second, the following types of curves are not allowed:

- curves that bound an unpunctured disk or a disk with a single (opened) puncture; and
- curves with two endpoints on the boundary of $\mathbf{S}_{\bar{\mathbf{M}}}$ which are isotopic to a piece of boundary containing no marked points, or a single marked point.

There is a natural projection map taking lifted measured laminations \bar{L} on $(\bar{\mathbf{S}}, \bar{\mathbf{M}})$ to unbounded measured laminations L on (\mathbf{S}, \mathbf{M}) : take the endpoints of \bar{L} that end at an opened puncture C_p and make them spiral around the corresponding puncture p in the direction *opposite* to the orientation chosen (in \bar{L}) on C_p . In this case \bar{L} is said to be a *lift* of L .

Since all curves in a lamination have a consistent direction of spiraling, every lamination has at least one lift.

This notion straightforwardly generalizes to multi-laminations: a *lifted multi-lamination* $\bar{\mathbf{L}}$ consists of an (uncoordinated) collection of lifted laminations. In particular, each of these lifted laminations has its own orientation on each C_p . The notion of projection likewise carries over.

Remark 14.3. For a lamination L on (\mathbf{S}, \mathbf{M}) with at least one curve spiraling into each puncture, the lifts \bar{L} of L are parametrized by $\mathbb{Z}^{\text{number of punctures}}$. These lifts differ by twisting \bar{L} around the opened circles C_p , as illustrated in Figure 39. This is analogous to Remark 10.3. On the other hand, when L has no spiraling into a given puncture p , the lifts \bar{L} have no endpoints on C_p —so locally, there are only two lifts, corresponding to the choices of orientation on C_p . To complete the analogy, we could forget the orientation here (in this case, it has no effect on the projection anyway), and admit curves in \bar{L} that enclose a single marked point (either a closed curve enclosing C_p , or a curve cutting off a single marked point on the boundary); these extra curves are the analogues of the choice of a horocycle. We do not pursue this further here since for our main goal (Theorem 13.6), we only need a single lift of any lamination, and since in order to get a full \mathbb{Z} 's worth of lifts and deal correctly with tagged arcs we would have to allow virtual (i.e., formally negative) curves enclosing a single marked point.

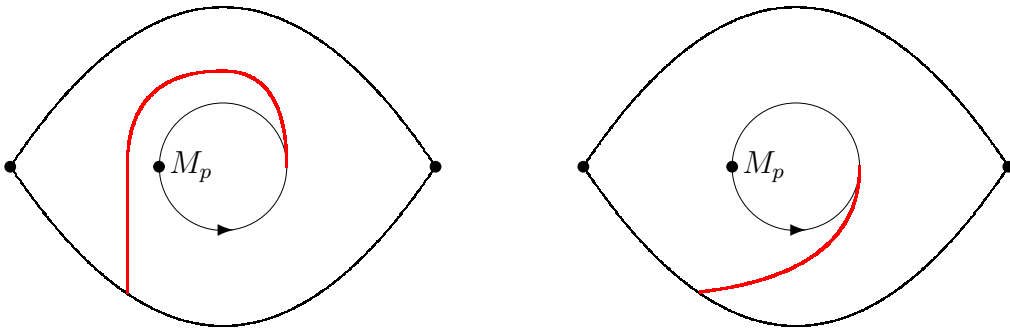


FIGURE 39. Different lifts of the lamination L_1 from Figure 36.

Definition 14.4 (*Transverse measures*). Let \bar{L} be a lifted lamination on the opened surface $(\bar{\mathbf{S}}, \bar{\mathbf{M}})$. Let γ be an (ordinary) arc in $\mathbf{A}^\circ(\bar{\mathbf{S}}, \bar{\mathbf{M}})$, or a boundary segment in $\mathbf{B}(\mathbf{S}, \mathbf{M})$. The *transverse measure* of γ with respect to \bar{L} is an integer denoted by $l_{\bar{L}}(\gamma)$ and defined as follows.

If γ does not have ends at M_p (for $p \in \overline{\mathbf{M}}$), then $l_{\overline{L}}(\gamma)$ is simply the (non-negative) minimal number of intersection points between the curves in \overline{L} and any arc homotopic to γ (relative to endpoints).

Next suppose γ has one or two ends that end at M_p for $p \in \overline{\mathbf{M}}$, and that γ has no notches. If γ twists sufficiently far around the corresponding opening(s) in the direction consistent with the orientation of \overline{L} at each end, then, again, $l_{\overline{L}}(\gamma)$ is equal to the minimal number of intersections between γ and \overline{L} . (The notion of “sufficiently far” will depend on the choice of a lift \overline{L} .)

We then extend the definition to all untagged arcs using the approach utilized earlier to define $l(\gamma)$ in Definition 10.6. By analogy with (10.2), we require that

$$(14.1) \quad l_{\overline{L}}(\psi_p \gamma) = n_p(\gamma) l_{\overline{L}}(p) + l_{\overline{L}}(\gamma),$$

where, as before, we use the notation (10.3)–(10.4), and

$$(14.2) \quad l_{\overline{L}}(p) = \begin{cases} -l_{\overline{L}}(C_p) & p \in P, \text{ if } C_p \text{ is oriented counterclockwise;} \\ 0 & \text{if } p \notin P; \\ l_{\overline{L}}(C_p) & p \in P, \text{ if } C_p \text{ is oriented clockwise.} \end{cases}$$

As in the earlier case, property (14.1) is consistent with the definition given above for the arcs that twist sufficiently far. Note that with this extended definition, the numbers $l_{\overline{L}}(\gamma)$ may be negative. See Figure 40.

Finally, for a tagged arc $\gamma \in \mathbf{A}^\infty(\mathbf{S}, \overline{\mathbf{M}})$ which is notched at p and twists sufficiently far in the direction *opposite* to the orientation of \overline{L} on C_p , define $l_{\overline{L}}(\gamma)$ to be the number of intersections of γ with \overline{L} , plus $|l_{\overline{L}}(p)|$. (This extra term corresponds to the asymptotics of $\nu(p)$ as described in Remark 10.12, and will make Lemma 14.10 below come out uniformly.) For notched arcs that do not twist sufficiently far, we extend the definition using (14.1); as before, $l_{\overline{L}}(\gamma)$ may be negative. (Recall that $n_p(\gamma) < 0$ if γ is notched at p .)

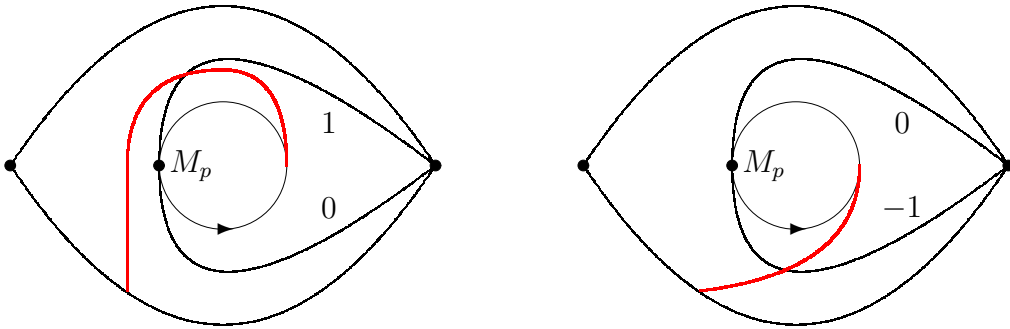


FIGURE 40. Transverse measures of arcs with respect to a lifted lamination. Here $l_{\overline{L}}(C_p) = 1$, so $l_{\overline{L}}(p) = -1$, and (14.1) becomes $l_{\overline{L}}(\psi_p \gamma) = l_{\overline{L}}(\gamma) - 1$.

We prefer to write these transverse measures multiplicatively, to make the analogy with geometric lambda lengths stronger and to match usual cluster algebra notation.

Definition 14.5 (*Tropical semifield associated with a multi-lamination*). Let $\mathbf{L} = (L_i)_{i \in I}$ be a multi-lamination in (\mathbf{S}, \mathbf{M}) ; here I is a finite indexing set. We introduce a formal variable q_i for each lamination L_i , and set (see Definition 4.1)

$$(14.3) \quad \mathbb{P}_{\mathbf{L}} = \text{Trop}(q_i : i \in I).$$

Definition 14.6 (*Tropical lambda lengths*). We continue in close analogy with Definition 10.6. Let $\overline{\mathbf{L}} = (\overline{L}_i)_{i \in I}$ be a lift of a multi-lamination \mathbf{L} . Instead of exponentiating the distances to get the lambda lengths (cf. (10.1)), we define the *tropical lambda length* of $\gamma \in \mathbf{A}^\infty(\overline{\mathbf{S}}, \overline{\mathbf{M}}) \cup \mathbf{B}(\mathbf{S}, \mathbf{M})$ with respect to $\overline{\mathbf{L}}$ by

$$(14.4) \quad c_{\overline{\mathbf{L}}}(\gamma) = \prod_{i \in I} q_i^{-l_{\overline{L}_i}(\gamma)/2} \in \mathbb{P}_{\mathbf{L}}.$$

If we set

$$(14.5) \quad c_{\overline{\mathbf{L}}}(p) = \prod_{i \in I} q_i^{-l_{\overline{L}_i}(p)/2} \in \mathbb{P}_{\mathbf{L}}$$

(cf. (10.7)), then by (10.12) the tropical lambda lengths satisfy

$$(14.6) \quad c_{\overline{\mathbf{L}}}(\psi_p \gamma) = c_{\overline{\mathbf{L}}}(p)^{n_p(\gamma)} c_{\overline{\mathbf{L}}}(\gamma).$$

Remark 14.7. The minus signs in the exponents of equations (14.4)-(14.5) are there because transformations of transverse measures involve maxima (as in (14.8) below) whereas the tropical semifield is defined using minima.

We can also define, in a similar way, the quantities $l_{\overline{\mathbf{L}}}(\gamma)$ and $c_{\overline{\mathbf{L}}}(\gamma)$ when γ is a tagged loop based at a marked point and enclosing a sole puncture/opening.

Remark 14.8. Our definition of a tropical lambda length of a (lifted) tagged arc depends on the choice of a lift $\overline{\mathbf{L}}$ of the multi-lamination \mathbf{L} . Different choices result in different notions of a tropical lambda length. However, they all differ from each other by *gauge transformations* which simultaneously rescale the lambda lengths of all arcs incident to a given puncture, and do not affect the resulting cluster algebra structure.

On the other hand, the tropical (or ordinary) lambda lengths of boundary segments or holes do not depend on the choice of a lift $\overline{\mathbf{L}}$. Consequently, we can use notation $c_{\mathbf{L}}(\beta) = c_{\overline{\mathbf{L}}}(\beta)$ for $\beta \in \mathbf{B}(\mathbf{S}, \mathbf{M})$, or $c_{\mathbf{L}}(p) = c_{\overline{\mathbf{L}}}(p)$ for $p \in \overline{\mathbf{M}}$. The (tropical) lambda lengths of arcs not incident to punctures are similarly independent on the choice of $\overline{\mathbf{L}}$.

It is perhaps worth emphasizing that all our lambda lengths, whether tropical or ordinary, do not depend on a (tagged) triangulation containing the (tagged) arc in question.

The main property of the tropical lambda lengths is that they satisfy the tropical version of the exchange relations (10.8) and (10.15).

Lemma 14.9. *On the opened surface $(\overline{\mathbf{S}}, \overline{\mathbf{M}})$, consider a quadrilateral with sides $\alpha, \beta, \gamma, \delta$ and diagonals η and θ , cf. Figure 12. Assume that the tagging of the arcs*

in $\{\alpha, \beta, \gamma, \delta, \eta, \theta\}$ is consistent at each marked point. Then

$$(14.7) \quad c_{\bar{\mathbf{L}}}(\eta)c_{\bar{\mathbf{L}}}(\theta) = c_{\bar{\mathbf{L}}}(\alpha)c_{\bar{\mathbf{L}}}(\gamma) \oplus c_{\bar{\mathbf{L}}}(\beta)c_{\bar{\mathbf{L}}}(\delta),$$

where \oplus denotes the tropical addition in $\mathbb{P}_{\mathbf{L}}$.

Proof. It is immediate from the definition of the tropical semifield (see Definition 4.1) that it suffices to prove (14.7) in the case when $\bar{\mathbf{L}}$ consists of a single lamination \bar{L} . In that case, (14.7) becomes

$$(14.8) \quad l_{\bar{L}}(\theta) + l_{\bar{L}}(\eta) = \max(l_{\bar{L}}(\alpha) + l_{\bar{L}}(\gamma), l_{\bar{L}}(\beta) + l_{\bar{L}}(\delta)).$$

If all the arcs twist sufficiently far around the openings (if any) containing their ends, then $l_{\bar{L}}$ is an intersection number and this is a well known (and easy to check) relation (see, e.g., [7, Section 3]). The general case follows by noticing that (14.8) is invariant under twists around openings. \square

Lemmas 14.10 and 14.11 below are tropical analogues of Lemmas 10.14 and 10.15, respectively.

Lemma 14.10. *Consider an opened monogon as shown in Figure 27, with boundary marked point q and opened puncture C_p . If δ and ϱ are compatible parallel tagged arcs connecting q and M_p , one of them plain and one notched at M_p , then we have*

$$(14.9) \quad c_{\bar{\mathbf{L}}}(\delta)c_{\bar{\mathbf{L}}}(\varrho) = c_{\bar{\mathbf{L}}}(\eta).$$

Proof. Straightforward to check, as there are only three elementary laminations in a punctured monogon. \square

Lemma 14.11. *Consider an opened digon as shown in Figure 30. Under the assumptions of Lemma 10.15, we have*

$$(14.10) \quad c_{\bar{\mathbf{L}}}(\varrho)c_{\bar{\mathbf{L}}}(\theta) = c_{\bar{\mathbf{L}}}(\alpha) \oplus c_{\bar{\mathbf{L}}}(p)^{-1}c_{\bar{\mathbf{L}}}(\beta).$$

Proof. Just as Lemma 10.15 follows from Lemma 10.14, this follows straightforwardly from Lemma 14.10. Alternatively, this can be checked by examining the six elementary laminations on a punctured digon from Figure 36. \square

In the language of transverse measures (cf. (14.8)), the relation (14.10) corresponds to the identity

$$(14.11) \quad l_{\bar{L}}(\varrho) + l_{\bar{L}}(\theta) = \max(l_{\bar{L}}(\alpha), l_{\bar{L}}(\beta) - l_{\bar{L}}(p)).$$

We then obtain an analogue of Theorem 11.1.

Corollary 14.12. *Fix a multi-lamination \mathbf{L} and its lift $\bar{\mathbf{L}}$. For each $\gamma \in \mathbf{A}^{\bowtie}(\mathbf{S}, \mathbf{M})$, fix an arbitrary lift $\bar{\gamma}$. Then the tropical lambda lengths $\{c_{\bar{\mathbf{L}}}(\bar{\gamma})\}$ satisfy (3.3), the exchange relations in the tropical semifield $\mathbb{P}_{\mathbf{L}}$, for the same exchange matrices and the same choices of coefficients as in Theorem 11.1.*

We now relate the tropical shear coordinates from Sections 12 and 13 to lifted lambda lengths. The statement below is a version of a well known formula (see, e.g., [32, p. 44], [3, Section 4.6]) relating shear coordinates to transverse measures.

Lemma 14.13. *Let T be a triangulation of (\mathbf{S}, \mathbf{M}) , let L be a lamination on (\mathbf{S}, \mathbf{M}) , and let η be an arc in T that is not contained inside a self-folded triangle. Let $\alpha, \beta, \gamma, \delta$ be the arcs on the boundary of the quadrilateral containing η , arranged as in Figure 12, and let $\bar{\alpha}_0, \bar{\beta}_0, \bar{\gamma}_0, \bar{\delta}_0$ be compatible lifts of them to $(\bar{\mathbf{S}}, \bar{\mathbf{M}})$, in the sense that they form a quadrilateral on $(\bar{\mathbf{S}}, \bar{\mathbf{M}})$. Then for any lift \bar{L} of L , we have*

$$(14.12) \quad 2b_\eta(T, L) = -l_{\bar{L}}(\bar{\alpha}_0) + l_{\bar{L}}(\bar{\beta}_0) - l_{\bar{L}}(\bar{\gamma}_0) + l_{\bar{L}}(\bar{\delta}_0).$$

Note that some of $\alpha, \beta, \gamma, \delta$ may not be tagged arcs in the strict sense of Definition 5.7, as they may enclose a once-punctured monogon; still, $l_{\bar{L}}$ is well defined.

Proof. This is a consequence of Definition 12.2, as follows. Assume that \bar{L} is twisted sufficiently far in the sense of Definition 14.6. Each ‘S’-shaped intersection of \bar{L} with η contributes +2 to the right hand side, each ‘Z’-shaped intersection contributes -2, while intersections of \bar{L} with the quadrilateral that cut off a corner does not effect the right hand side. Changing the lift \bar{L} of L effectively changes the number of intersections that cut off corners. In particular, we can modify \bar{L} so that it twists sufficiently far without changing the right hand side. \square

Lemma 14.14. *For T a tagged triangulation of (\mathbf{S}, \mathbf{M}) and $\eta \in T$ a tagged arc that is not parallel to any other arc, let $\alpha, \beta, \gamma, \delta$ be the tagged arcs on the boundary of the quadrilateral containing η , arranged as in Figure 12. (Some or all of α, \dots, δ may be curves enclosing punctured monogons and not in T itself.) Let $\bar{\alpha}_0, \bar{\beta}_0, \bar{\gamma}_0, \bar{\delta}_0$ be compatible lifts to $(\bar{\mathbf{S}}, \bar{\mathbf{M}})$. Then for any lift \bar{L} of L , (14.12) holds.*

Proof. This is a consequence of Lemma 14.13 and Part (i) of Definition 13.1, as follows. Let L_1 and L_2 be two laminations on (\mathbf{S}, \mathbf{M}) that differ only in the direction of spiraling at a puncture p . Then, if \bar{L}_1 is a lift of L_1 , we can find a lift \bar{L}_2 of L_2 by simply changing the orientation of C_p , without changing the actual curves corresponding to \bar{L}_1 in $(\bar{\mathbf{S}}, \bar{\mathbf{M}})$. Then, for any two lifted arcs $\bar{\alpha}_1$ and $\bar{\alpha}_2$ which are identical except that $\bar{\alpha}_1$ is plain at p and $\bar{\alpha}_2$ is notched at p , by Definition 14.4 we have

$$(14.13) \quad l_{\bar{L}_1}(\bar{\alpha}_2) = l_{\bar{L}_2}(\bar{\alpha}_1) + |l_{\bar{L}_1}(p)|,$$

as when $\bar{\alpha}_1$ is twisted sufficiently far with respect to \bar{L}_1 , $\bar{\alpha}_2$ is twisted sufficiently far with respect to \bar{L}_2 , as both are trying to twist in the same direction, $\bar{\alpha}_1$ with the orientation of C_p and $\bar{\alpha}_2$ against the reversed orientation of C_p .

Thus, for each vertex of the quadrilateral at which there are notches, we can simultaneously remove them and reverse the direction of spiraling of L without changing the sum on the right of (14.12), in accordance with Part (i) of Definition 13.1. \square

Lemma 14.15. *Let T be a tagged triangulation of (\mathbf{S}, \mathbf{M}) , L be a lamination on (\mathbf{S}, \mathbf{M}) , and let ϱ and γ be parallel arcs in T that differ in tagging at one end, call it p . The arcs ϱ and γ are contained inside a digon in T where the other sides are labeled α and β , as in Figure 41. Let $\bar{\alpha}_0$ and $\bar{\beta}_0$ be compatible lifts of α and β to $(\bar{\mathbf{S}}, \bar{\mathbf{M}})$. Then for any lift \bar{L} of L ,*

$$(14.14) \quad 2b_\varrho(T, L) = -l_{\bar{L}}(\bar{\alpha}_0) + l_{\bar{L}}(\bar{\beta}_0) + n_p(\varrho)l_{\bar{L}}(p).$$

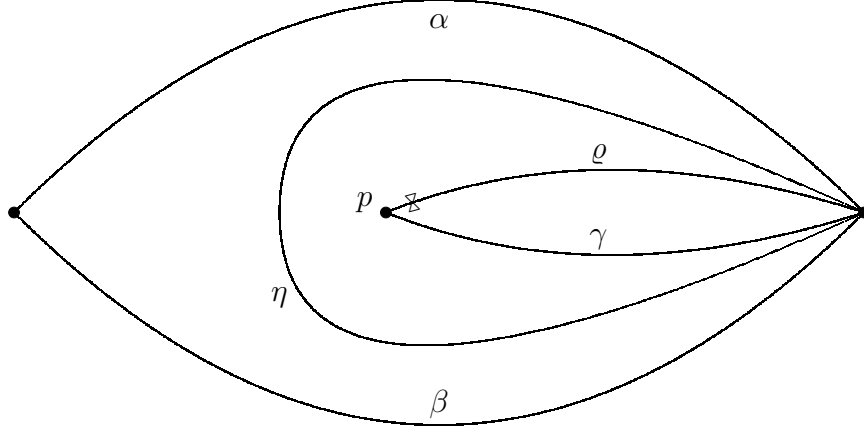


FIGURE 41. Proof of Lemma 14.15.

As before, α and β may enclose a once-punctured monogon and so not be tagged arcs.

Proof. Suppose that ρ is notched at p , and suppose there is an ordinary triangulation T° so that $\tau(T^\circ) = T$. Let η be the arc that projects to ρ . Then by Part (ii) of Definition 13.1, $b_\rho(T, L) = b_\eta(T^\circ, L)$. Now by Lemma 14.13,

$$(14.15) \quad b_\eta(T^\circ, L) = -l_{\overline{L}}(\overline{\alpha}_0) + l_{\overline{L}}(\overline{\beta}_0) - l_{\overline{L}}(\overline{\gamma}_0) + l_{\overline{L}}(\overline{\delta}_0),$$

where $\overline{\alpha}_0, \overline{\beta}_0, \overline{\gamma}_0$, and $\overline{\delta}_0 = \psi_p(\overline{\gamma}_0)$ are compatible lifts of α, β, γ and again γ . The result then follows by equation (14.1).

If the tags differ from the case above, then the result follows by applying tag-changing transformations and changing the spiraling of L , as in Lemma 14.14. \square

We can now give a conceptual proof that tropical shear coordinates behave as expected under mutation. For any triangulation T (ordinary or tagged), multi-lamination \mathbf{L} , and arc $\eta \in T$, define

$$(14.16) \quad r_\eta(T, \mathbf{L}) = \prod_{i \in I} q_i^{-b_\eta(T, L_i)}.$$

Proof of Theorem 12.6. Fix an ordinary triangulation T without self-folded triangles and multi-lamination \mathbf{L} with lift $\overline{\mathbf{L}}$. For each arc $\gamma \in T$, fix a lift $\overline{\gamma}$. In the r variables, (14.12) becomes

$$(14.17) \quad r_\eta(T, \mathbf{L}) = c_{\overline{\mathbf{L}}}(\overline{\alpha}_0)^{-1} \cdot c_{\overline{\mathbf{L}}}(\overline{\beta}_0) \cdot c_{\overline{\mathbf{L}}}(\overline{\gamma}_0)^{-1} \cdot c_{\overline{\mathbf{L}}}(\overline{\delta}_0).$$

Rewriting equation (14.17) in terms of our chosen lifts $\overline{\alpha}, \dots, \overline{\delta}$ (which are not necessarily compatible) and the exchange matrix $B(T)$, we obtain

$$(14.18) \quad r_\eta(T, \mathbf{L}) = \frac{p_\eta^+}{p_\eta^-} \prod_{\theta \in T} c_{\overline{\mathbf{L}}}(\overline{\theta})^{B(T)_{\theta\eta}},$$

where the product runs over all edges $\theta \in T$, and p_η^\pm are the coefficients from Theorem 11.1. (An edge θ in T that is not adjacent to η will not contribute to the product,

as then $B(T)_{\theta\eta} = 0$. The factor p_η^+/p_η^- makes up for taking the lifts $\bar{\alpha}, \dots, \bar{\delta}$ rather than $\bar{\alpha}_0, \dots, \bar{\delta}_0$.)

Thus the variables $r_\eta(T, \mathbf{L})$ are defined just like the \hat{y} variables in (2.14), but with the variables $c_{\mathbf{L}}$. By Corollary 14.12, the $c_{\mathbf{L}}$ transform as cluster variables in the tropical semifield $\mathbb{P}_{\mathbf{L}}$. Thus Proposition 2.9 applies, and the $r_\eta(T, \mathbf{L})$ transform according to equation (2.15) (interpreted tropically). As noted in Remark 2.10, this is how the coefficients in the B matrix transform, as claimed in the statement of Theorem 12.6. \square

Proof of Theorem 13.5. For the more general setting of Theorem 13.5, we must extend the above arguments to the case of tagged triangulations. As in Definition 13.1, there is an untagged triangulation T° so that T is obtained from T° by applying tag-changing transformations to $\tau(T^\circ)$. An arc $\eta \in T$ can have a parallel copy with different tagging (in which case the corresponding arc $\eta^\circ \in T^\circ$ is part of a self-folded triangle), or not. In the second case, by Lemma 14.14, (14.17) holds (where, as before, $\bar{\alpha}_0, \dots, \bar{\delta}_0$ are compatible lifts of the quadrilateral containing η). If one of α, \dots, δ is a tagged curve enclosing a once-punctured monogon, Lemma 14.10 applies. This combines with the behaviour of $B(T)$ in the case of self-folded triangles (see [9, Definitions 4.1 and 9.7]) to show that (14.18) is true in this case as well. (As before, the factor p_η^+/p_η^- makes up for taking the original lifts rather than compatible lifts.)

On the other hand, let $\varrho \in T$ be a tagged arc with a parallel copy γ . Then we can apply Lemma 14.15 to conclude

$$(14.19) \quad r_\varrho(T, \mathbf{L}) = c_{\mathbf{L}}(\bar{\alpha}_0)^{-1} \cdot c_{\mathbf{L}}(\bar{\beta}_0) \cdot c_{\mathbf{L}}(p)^{n_p(\varrho)}.$$

But this is yet another form of (14.18) for the arc ϱ . (The last factor in (14.19) becomes part of the coefficient factor p_ϱ^+/p_ϱ^- .)

Thus for all arcs in T , (14.18) holds and the r_η are defined like the \hat{y} variables with respect to the tropical lambda lengths $c_{\mathbf{L}}$. The result follows by Remark 2.10. \square

15. LAMINATED TEICHMÜLLER SPACES

In this section, we use the notions developed in previous sections—specifically, ordinary and tropical lambda lengths of tagged arcs on an opened surface—to present our main construction, a geometric realization of cluster algebras associated with surfaces. The main idea is rather natural. As the lifts $\bar{\gamma}$ of an arc γ vary, the corresponding tropical lambda lengths, just like the ordinary ones, form a geometric progression (cf. (10.12) and (14.6)). After making sure that the ratios of the two progressions (ordinary and tropical) coincide, we proceed by dividing an ordinary (rescaled) lambda length of $\bar{\gamma}$ by a tropical one, thus obtaining an invariant of γ that plays the role of a cluster variable.

Definition 15.1 (*Laminated Teichmüller space*). Let $\mathbf{L} = (L_i)_{i \in I}$ be a multi-lamination in (\mathbf{S}, \mathbf{M}) . The *laminated Teichmüller space* $\bar{\mathcal{T}}(\mathbf{S}, \mathbf{M}, \mathbf{L})$ is defined as follows. A point $(\sigma, q) \in \bar{\mathcal{T}}(\mathbf{S}, \mathbf{M}, \mathbf{L})$ is a decorated hyperbolic structure $\sigma \in \bar{\mathcal{T}}(\mathbf{S}, \mathbf{M})$ together

with a collection of positive real weights $q = (q_i)_{i \in I}$ which are chosen so that the following boundary conditions hold:

- for each boundary segment $\beta \in \mathbf{B}(\mathbf{S}, \mathbf{M})$, we have $\lambda_\sigma(\beta) = c_{\mathbf{L}}(\beta)$;
- for each hole C_p , with $p \in \overline{\mathbf{M}}$, we have $\lambda_\sigma(p) = c_{\mathbf{L}}(p)$.

In these equations, the quantities $c_{\mathbf{L}}(\beta)$ and $c_{\mathbf{L}}(p)$ (cf. Remark 14.8) are given by the formulas (14.4) and (14.5), with each q_i specialized to the given positive real value. In more concrete terms, we require that for each boundary segment β ,

$$(15.1) \quad l_\sigma(\beta) = - \sum_{i \in I} l_{\overline{\mathbf{L}}_i}(\beta) \ln(q_i),$$

and similarly with β replaced by p . Informally, our boundary conditions (15.1) ask that for each hole (resp., boundary segment), the total weighted sum of its transverse measures with respect to the laminations in \mathbf{L} is the negative of the ordinary hyperbolic length, with respect to σ , of the hole (resp., segment between horocycles).

We next coordinatize the laminated Teichmüller space $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M}, \mathbf{L})$. A system of coordinates will include the weights of laminations plus the lambda lengths of the arcs in a (tagged) triangulation.

Proposition 15.2. *Let $\mathbf{L} = (L_i)_{i \in I}$ be a multi-lamination in (\mathbf{S}, \mathbf{M}) . The laminated Teichmüller space $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M}, \mathbf{L})$ can be coordinatized as follows. Fix an ideal (or tagged) triangulation T of (\mathbf{S}, \mathbf{M}) . Choose a lift of each of the n arcs $\gamma \in T$ to an arc $\overline{\gamma} \in \mathbf{A}^\circ(\overline{\mathbf{S}}, \overline{\mathbf{M}})$. Then the map*

$$\Psi : \overline{\mathcal{T}}(\mathbf{S}, \mathbf{M}, \mathbf{L}) \rightarrow \mathbb{R}_{>0}^{n+|I|}$$

defined by

$$\Psi(\sigma, q) = (\lambda_\sigma(\overline{\gamma}))_{\gamma \in T} \times q$$

is a homeomorphism.

Proof. This is a direct consequence of Proposition 10.10 and Definition 15.1. \square

Proposition 15.2 enables us to view the coordinate functions q_i and $\lambda(\overline{\gamma})$ as “variables,” similarly to our treatment of the earlier versions of the Teichmüller space.

We next define the quantities that will play the role of cluster variables in our main construction.

Definition 15.3 (*Laminated lambda lengths*). Fix a lift $\overline{\mathbf{L}}$ of a multi-lamination \mathbf{L} . For a tagged arc $\gamma \in \mathbf{A}^\circ(\mathbf{S}, \mathbf{M})$, the *laminated lambda length* $x_{\overline{\mathbf{L}}}(\gamma)$ is a function on the laminated Teichmüller space $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M}, \mathbf{L})$ defined by

$$(15.2) \quad x_{\overline{\mathbf{L}}}(\gamma) = \lambda(\overline{\gamma}) / c_{\overline{\mathbf{L}}}(\overline{\gamma}),$$

where $\overline{\gamma}$ is an arbitrary lift of γ and $c_{\overline{\mathbf{L}}}(\overline{\gamma})$ is the tropical lambda length of Definition 14.6. The value of $x_{\overline{\mathbf{L}}}(\gamma)$ does not depend on the choice of the lift $\overline{\gamma}$ since $\lambda(\overline{\gamma})$ and $c_{\overline{\mathbf{L}}}(\overline{\gamma})$ rescale by the same factor $c_{\mathbf{L}}(p)^{\pm 1} = \lambda_\sigma(p)^{\pm 1}$ (see Definition 15.1) as $\overline{\gamma}$ twists around the opening C_p (cf. (10.12) and (14.6)).

On the other hand, $x_{\overline{\mathbf{L}}}(\gamma)$ does depend on the choice of the lifted multi-lamination $\overline{\mathbf{L}}$. This will not create problems as the resulting cluster structure will be unique up to gauge transformations (see Remark 14.8), hence up to a canonical isomorphism.

Remark 15.4. The laminated lambda lengths $x_{\overline{\mathbf{L}}}(\gamma)$ can be intuitively interpreted as follows. Suppose that the lift $\overline{\gamma}$ is such that it twists sufficiently far around each of its ends lying on opened components. (Since some of the \overline{L}_i may have opposite orientation on a given C_p , it may be impossible to satisfy this condition for all i simultaneously, as $\overline{\gamma}$ may be required to spiral in opposite directions; still, let us assume that we can.) Then we combine the definition (15.2) with (10.1) and (14.4) to get

$$(15.3) \quad x_{\overline{\mathbf{L}}}(\gamma) = e^{l(\overline{\gamma})/2} \prod_{i \in I} q_i^{l_{\overline{L}_i}(\overline{\gamma})/2}.$$

Thus $x_{\overline{\mathbf{L}}}(\gamma)$ is obtained by exponentiating a sum of two kinds of contributions:

- the hyperbolic length of the lifted arc $\overline{\gamma}$ between appropriate horocycles (“the cost of fuel while traveling along $\overline{\gamma}$ ”), and
- a fixed contribution, depending on the q_i , associated with each crossing of \overline{L}_i by $\overline{\gamma}$ (“the tolls”).

Corollary 15.5. *Let $\mathbf{L} = (L_i)_{i \in I}$ be a multi-lamination in (\mathbf{S}, \mathbf{M}) . For a tagged triangulation T and any choice of lift $\overline{\mathbf{L}}$ of \mathbf{L} , the map*

$$\begin{aligned} \overline{\mathcal{T}}(\mathbf{S}, \mathbf{M}, \mathbf{L}) &\rightarrow \mathbb{R}_{>0}^{n+|I|} \\ (\sigma, q) &\mapsto (x_{\overline{\mathbf{L}}}(\gamma))_{\gamma \in T} \times q \end{aligned}$$

is a homeomorphism.

Proof. This is a corollary of Proposition 15.2 and Definition 15.3. \square

We are finally prepared to present our main construction.

Theorem 15.6. *For a given multi-lamination $\mathbf{L} = (L_i)_{i \in I}$, there exists a unique normalized exchange pattern (Σ_T) of geometric type with the following properties:*

- the coefficient semifield is the tropical semifield $\mathbb{P}_{\mathbf{L}} = \text{Trop}(q_i : i \in I)$;
- the cluster variables $x_{\mathbf{L}}(\gamma)$ are labeled by the tagged arcs $\gamma \in \mathbf{A}^{\bowtie}(\mathbf{S}, \mathbf{M})$;
- the seeds $\Sigma_T = (\mathbf{x}(T), \tilde{B}(T, \mathbf{L}))$ are labeled by the tagged triangulations T ;
- the exchange graph is the graph $\mathbf{E}(\mathbf{S}, \mathbf{M})$ of tagged flips (see Definition 5.14);
- each cluster $\mathbf{x}(T)$ consists of cluster variables $x_{\mathbf{L}}(\gamma)$, for $\gamma \in T$;
- the ambient field \mathcal{F} is generated over $\mathbb{P}_{\mathbf{L}}$ by any given cluster $\mathbf{x}(T)$;
- the extended exchange matrix $\tilde{B}(T, \mathbf{L})$ is described in Definition 13.3.

This exchange pattern has a positive realization (see Definition 4.4) by functions on the laminated Teichmüller space $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M}, \mathbf{L})$. To obtain this realization, choose a lift $\overline{\mathbf{L}}$ of the multi-lamination \mathbf{L} ; then represent each cluster variable $x_{\mathbf{L}}(\gamma)$ by the corresponding laminated lambda length $x_{\overline{\mathbf{L}}}(\gamma)$, and each coefficient variable q_i by the corresponding function on $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M}, \mathbf{L})$.

Proof. We already know (see Theorem 13.5) that the matrices $\tilde{B}(T, \mathbf{L})$ are related to each other by mutations associated with the tagged flips. In view of Propositions 4.5 and 15.2, all we need to prove is that the laminated lambda lengths $x_{\mathbf{L}}(\gamma)$ satisfy the exchange relations encoded by the matrices $\tilde{B}(T, \mathbf{L})$.

Fix arbitrary lifts $\bar{\gamma}$ of all tagged arcs $\gamma \in \mathbf{A}^{\text{tag}}(\mathbf{S}, \mathbf{M})$. By Theorem 11.1, the lambda lengths of lifted arcs $x(\bar{\gamma})$ form a (non-normalized) exchange pattern on $\mathbf{E}(\mathbf{S}, \mathbf{M})$. This statement remains true if we view $x(\bar{\gamma})$ as a function on $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M}, \mathbf{L})$ rather than $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M})$, replacing the coefficient group $\mathbb{P}(\mathbf{S}, \mathbf{M})$ (see (11.1)) by $\text{Trop}(q_i)$. Indeed, the monomial mutation rules (2.2)–(2.3) are satisfied by the coefficients in exchange relations for $x(\bar{\gamma})$'s viewed as functions on $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M})$, and therefore they are satisfied after the lambda lengths of boundary segments and holes are replaced by monomials in the lamination weights.

By Lemmas 14.9 and 14.11, their tropical counterparts $c_{\mathbf{L}}(\bar{\gamma})$ satisfy the tropical versions of the same exchange relations. By Proposition 3.4, this is exactly what is needed in order for the rescaled lambda lengths $x(\bar{\gamma})/c_{\mathbf{L}}(\bar{\gamma})$ to form a normalized exchange pattern.

It remains to verify that the extended exchange matrices of this exchange pattern are the matrices $\tilde{B}(T, \mathbf{L})$. (This in itself implies the uniqueness statement in the theorem.) In fact, by Theorem 13.5, it suffices to do this for *some* triangulation, say an ordinary triangulation T without self-folded triangles. In this setting, we need to verify that the coefficients of the exchange relations associated with the flips from T (recall that these coefficients are given by (3.1)) coincide with the coefficients encoded by $\tilde{B}(T, \mathbf{L})$.

These exchange relations are obtained by dividing the Ptolemy relations (10.8) by their tropical counterparts (14.7). If arcs $\alpha, \beta, \gamma, \delta \in T$ form a quadrilateral with diagonals $\eta \in T$ and θ , as in Figure 12, then the corresponding exchange relation is

$$x_{\mathbf{L}}(\eta) x_{\mathbf{L}}(\theta) = p_{\eta}^{+} x_{\mathbf{L}}(\alpha) x_{\mathbf{L}}(\gamma) + p_{\eta}^{-} x_{\mathbf{L}}(\beta) x_{\mathbf{L}}(\delta),$$

where

$$(15.4) \quad p_{\eta}^{+} = \frac{c_{\mathbf{L}}(\bar{\alpha}) c_{\mathbf{L}}(\bar{\gamma})}{c_{\mathbf{L}}(\bar{\eta}) c_{\mathbf{L}}(\bar{\theta})}, \quad p_{\eta}^{-} = \frac{c_{\mathbf{L}}(\bar{\beta}) c_{\mathbf{L}}(\bar{\delta})}{c_{\mathbf{L}}(\bar{\eta}) c_{\mathbf{L}}(\bar{\theta})},$$

and $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}, \bar{\eta}, \bar{\theta}$ are lifts of the corresponding arcs in T , coordinated so that $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ form a quadrilateral with diagonals $\bar{\eta}$ and $\bar{\theta}$. If, for example, α is a boundary segment rather than an arc, then $x(\alpha) = \lambda(\alpha) = c_{\mathbf{L}}(\alpha) = c_{\mathbf{L}}(\alpha)$ (see Definition 15.1), and the exchange relation becomes

$$x_{\mathbf{L}}(\eta) x_{\mathbf{L}}(\theta) = p_{\eta}^{+} x_{\mathbf{L}}(\gamma) + p_{\eta}^{-} x_{\mathbf{L}}(\beta) x_{\mathbf{L}}(\delta),$$

with the coefficients p_{η}^{\pm} given by (15.4) as before. Since these coefficients satisfy the normalization condition $p_{\eta}^{+} \oplus p_{\eta}^{-} = 1$ (by Proposition 3.4, or by (14.7)), the claim reduces to showing that their ratio coincides with the Laurent monomial encoded by the appropriate column of the matrix $\tilde{B}(T, \mathbf{L})$. That is, we need to check that

$$\frac{p_{\eta}^{+}}{p_{\eta}^{-}} = \frac{c_{\mathbf{L}}(\bar{\alpha}) c_{\mathbf{L}}(\bar{\gamma})}{c_{\mathbf{L}}(\bar{\beta}) c_{\mathbf{L}}(\bar{\delta})} = \prod_{i \in I} q_i^{b_{\eta}(T, L_i)},$$

where, as in (12.1), $b_\eta(T, L_i)$ denotes the shear coordinate of the arc η with respect to the lamination L_i . In view of (14.4), this is equivalent to (14.12).

The latter is a version of a well known formula (see, e.g., [32, p. 44], [3, Section 4.6]) relating shear coordinates to transverse measures. \square

Definition 15.7. Let \mathbf{L} be a multi-lamination on (\mathbf{S}, \mathbf{M}) , a bordered surface with marked points. Then the cluster algebra of geometric type described in Theorem 15.6 is denoted by $\mathcal{A}(\mathbf{S}, \mathbf{M}, \mathbf{L})$.

Remark 15.8. The exchange pattern in Theorem 8.6—with the coefficient variables corresponding to the boundary segments in $\mathbf{B}(\mathbf{S}, \mathbf{M})$ —can be obtained as a particular case of the main construction of this section. The corresponding multi-lamination $\mathbf{L} = \{L_\beta\}_{\beta \in \mathbf{B}(\mathbf{S}, \mathbf{M})}$ contains one lamination L_β for each boundary segment $\beta \in \mathbf{B}(\mathbf{S}, \mathbf{M})$. The lamination L_β consists of a single curve (also denoted by L_β) defined as follows. Let $p, q \in \mathbf{M}$ be the endpoints of β . If $p = q$ (so that β lies on a boundary component with a single marked point), then L_β is a closed curve in \mathbf{S} encircling β . Otherwise, let p' and q' be two points on $\partial\mathbf{S} \setminus \beta$ located very close to p and q , respectively. Then L_β connects p' and q' within a small neighborhood of β .

See Example 16.1 for a particular instance of this construction.

Proof of Theorem 6.1. Uniqueness of an exchange pattern described in the theorem is clear: the mutation rules determine all seeds uniquely from the initial one. To prove existence, we need to show that a sequence of mutations that returns us back to the same tagged triangulation recovers the original seed. To use the terminology of [14, Definition 4.5], we want to show that the exchange graph of our pattern is *covered* by (and in fact is identical to) the graph $\mathbf{E}(\mathbf{S}, \mathbf{M})$ of tagged flips.

We first reduce this claim to the case of patterns of geometric type. Indeed, [14, Theorem 4.6] asserts that the exchange graph of any cluster algebra \mathcal{A} is covered by the exchange graph of a particular cluster algebra of geometric type, namely one that has the same exchange matrices as \mathcal{A} , and whose coefficients are *principal* at an arbitrarily chosen seed. (See [14, Definition 3.1] or Definition 17.1 below.)

For an exchange pattern of geometric type, we produce a positive realization (cf. Definition 4.4) using the construction of Theorem 15.6. The key role in the argument is played by our generalization of Thurston's coordinatization theorem (Theorems 12.3 and 13.6), which guarantees that we can get any coefficients (of geometric type) by making an appropriate (unique) choice of a multi-lamination \mathbf{L} .

It remains to show that all laminated lambda lengths $x_{\overline{\mathbf{L}}}(\gamma)$, for $\overline{\gamma} \in \mathbf{A}^\circ(\mathbf{S}, \mathbf{M})$, are distinct. There are several ways to prove this; here we give a sketch of one such proof. It is easy to see that if in some normalized cluster algebra, a particular sequence of mutations yields a seed containing a cluster variable equal to one of the cluster variables of the initial cluster, then the same phenomenon must hold in the cluster algebra defined by the same initial exchange matrix over the one-element semifield $\{1\}$. It is therefore enough to check the claim in the case of trivial coefficients. In our setting, this case can be viewed as a special instance of the construction presented in Theorem 8.6, with the length of each boundary segment

equal to 1. In that instance, the cluster variables $x(\gamma)$ are the lambda lengths of certain (pairwise non-isotopic) geodesics between appropriate horocycles.

We can use the torus action associated with changing the horocycles to show that we only need to consider pairs of tagged arcs connecting the same horocycles (around the same marked points).

So suppose we have two arcs, γ_1 and γ_2 , connecting the same pair of marked points p and q . If $p \neq q$, consider a loop ℓ that surrounds γ_1 , and consider a family of hyperbolic metrics that pinch at ℓ (i.e., the length of ℓ goes to 0). Then the length of γ_2 goes to ∞ whereas the family can be chosen to keep the length of γ_1 bounded.

If $p = q$, consider instead the two loops ℓ_1 and ℓ_2 obtained by pushing γ_1 around the puncture p on the two different sides. Again, consider a family of hyperbolic metrics that pinch at ℓ_1 and ℓ_2 . In this family, the length of γ_2 goes to ∞ while the length of γ_1 can remain bounded.

(We thank Y. Eliashberg for suggesting this last argument.) \square

Proof of Corollary 6.2. By Theorem 6.1, the cluster variables in \mathcal{A} are in bijection with tagged arcs, and the seeds with tagged triangulations. The claims in the Corollary then follow from the basic properties of the tagged arc complex (see [9, Section 7]). \square

16. TOPOLOGICAL REALIZATIONS OF SOME COORDINATE RINGS

The main construction of Section 15 can be used to produce topological realizations of (well known) cluster structures in coordinate rings of various algebraic varieties. Several such examples are presented in this section.

In each of these examples, the corresponding generalized decorated Teichmüller space $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M}, \mathbf{L})$ is naturally interpreted as the *totally positive* part of the corresponding *cluster variety* X , the spectrum of the associated cluster algebra. In other words, $\overline{\mathcal{T}}(\mathbf{S}, \mathbf{M}, \mathbf{L})$ can be identified with the set of those points in X at which all cluster and coefficient variables (equivalently, those belonging to a given extended cluster) take positive values. In each case, one recovers the usual notion of total positivity of matrices or its well known extensions to Grassmannians and other G/P 's (see, e.g., [10, 11, 25] and references therein).

Example 16.1 (*Grassmannians* $\text{Gr}_{2,n+3}(\mathbb{C})$). We already considered this cluster algebra in Example 8.10. The homogeneous coordinate ring of the Grassmannian $\text{Gr}_{2,n+3}(\mathbb{C})$ (with respect to its Plücker embedding) is generated by the Plücker coordinates Δ_{ij} , for $1 \leq i < j \leq n+3$, subject to the Grassmann-Plücker relations

$$\Delta_{ik} \Delta_{jl} = \Delta_{ij} \Delta_{kl} + \Delta_{il} \Delta_{jk} \quad (i < j < k < l).$$

These relations can be viewed as exchange relations in the cluster algebra $\mathcal{A}_n = \mathbb{C}[\text{Gr}_{2,n+3}]$ of cluster type A_n . This cluster algebra has $n+3$ coefficient variables

$$\Delta_{12}, \Delta_{23}, \dots, \Delta_{n+2,n+3}, \Delta_{1,n+3},$$

naturally corresponding to the sides of a convex $(n+3)$ -gon. The remaining $\frac{n(n+3)}{2}$ Plücker coordinates Δ_{ij} are the cluster variables; they are naturally labeled by the diagonals of the $(n+3)$ -gon. Since the exchange relations in \mathcal{A}_n can be viewed as

Ptolemy relations between the lambda lengths of sides and diagonals of an $(n+3)$ -gon (=unpunctured disk with $n+3$ marked points on the boundary), we find ourselves in a situation described in Remark 15.8, and can apply the construction discussed therein. The case $n=3$ is illustrated in Figure 42.

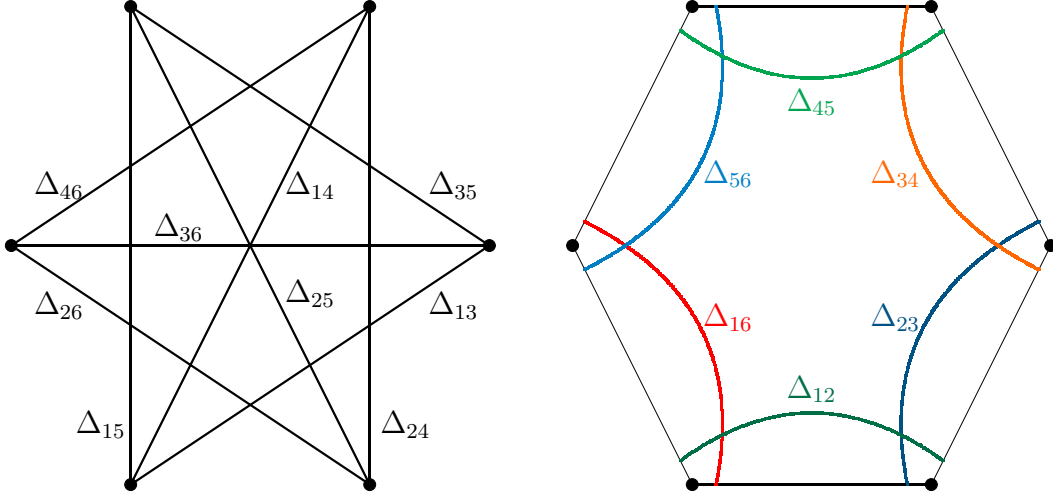


FIGURE 42. Representing the cluster structure on the Grassmannian $\text{Gr}_{2,6}$ by a multi-lamination of a hexagon.

Remark 16.2. We can also give a geometric correspondence between the Grassmannian $\text{Gr}_{2,n+3}(\mathbb{R})$ and the moduli space of decorated ideal $(n+3)$ -gons as in Example 8.10, as follows. The vector space of 2×2 symmetric matrices $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ carries a natural quadratic form

$$-\det M = b^2 - ac.$$

This quadratic form has signature $(2, 1)$ and is invariant under the action of $\text{SL}_2(\mathbb{R})$ by $A \cdot M = A^T M A$. Consider the subvariety

$$H = \left\{ \begin{pmatrix} a & b \\ b & c \end{pmatrix} : ac - b^2 = 1, a > 0 \right\}$$

of positive definite symmetric matrices with determinant 1. If we restrict $-\det(\cdot)$ to the tangent space to H at some point, we get a positive definite form. This gives H the structure of a Riemannian manifold, which is none other than the standard hyperboloid model of the hyperbolic plane \mathbb{H}^2 . The ideal boundary of \mathbb{H}^2 is the set of positive *null rays*, i.e., the set of rank 1, positive semidefinite symmetric matrices, considered up to scale. A decorated $(n+3)$ -gon is a choice of $n+3$ ideal points on the boundary of \mathbb{H}^2 together with a choice of horocycle around each ideal point, modulo symmetries of \mathbb{H}^2 . The choice of horocycle is the same as a choice of positive vector on the ideal ray, so a decorated $(n+3)$ -gon is a choice of $n+3$ rank 1, positive semi-definite matrices, up to simultaneous action by $\text{SL}_2(\mathbb{R})$ (but not scaling).

A generic point in $\text{Gr}_{2,n+3}(\mathbb{R})$ can be thought of as a sequence of $n+3$ vectors $\begin{pmatrix} p_i \\ q_i \end{pmatrix} \in \mathbb{R}^2$ modulo the action of $\text{SL}_2(\mathbb{R})$. The map

$$\begin{pmatrix} p_i \\ q_i \end{pmatrix} \mapsto w_i = \begin{pmatrix} p_i^2 & p_i q_i \\ p_i q_i & q_i^2 \end{pmatrix}$$

identifies such a sequence with a collection (w_i) of rank 1 positive semi-definite matrices. To establish the dictionary between respective coordinates, recall [26] that the λ -length between two null vectors w_i, w_j in the hyperboloid model can be defined as $\lambda(w_i, w_j) = \sqrt{-2\langle w_i, w_j \rangle}$, where $\langle \cdot, \cdot \rangle$ denotes the polarization of the quadratic form (the negated determinant). Elementary computation shows that

$$\langle w_i, w_j \rangle = -\frac{1}{2} \left(\det \begin{pmatrix} p_i & p_j \\ q_i & q_j \end{pmatrix} \right)^2,$$

implying that the lambda lengths $\lambda(w_i, w_j)$ and the Plücker coordinates Δ_{ij} agree up to sign. If the vectors in \mathbb{R}^2 are in one half-space and cyclically ordered, then all the Δ_{ij} are positive and the coordinates agree on the nose. The correct way to get the signs to match in general is to consider twisted systems of vectors; see [5, Section 11.1].

Example 16.3 (*Grassmannian $\text{Gr}_{3,6}$; cf. [29]*). The homogeneous coordinate ring $\mathcal{A} = \mathbb{C}[\text{Gr}_{3,6}]$ has a natural structure of a cluster algebra of (cluster) type D_4 , described in detail by J. Scott [29]. As a ring, \mathcal{A} is generated by the Plücker coordinates Δ_{ijk} , for $1 \leq i < j < k \leq 6$. As a cluster algebra, \mathcal{A} has 16 cluster variables, which include the cluster

$$(16.1) \quad \mathbf{x} = (x_1, x_2, x_3, x_4) = (\Delta_{245}, \Delta_{256}, \Delta_{125}, \Delta_{235}),$$

which we will use as an initial cluster. The cluster algebra \mathcal{A} has 6 coefficient variables

$$(x_5, \dots, x_{10}) = (\Delta_{123}, \Delta_{234}, \Delta_{345}, \Delta_{456}, \Delta_{156}, \Delta_{126}),$$

which generate its ground ring. The exchange relations from the initial seed, and the corresponding extended exchange matrix, with rows labeled by the cluster variables (rows 1–4) and coefficient variables (rows 5–10), are:

$$\begin{array}{l} \Delta_{245} \Delta_{356} = \Delta_{345} \Delta_{256} + \Delta_{456} \Delta_{235} \\ \Delta_{256} \Delta_{145} = \Delta_{456} \Delta_{125} + \Delta_{156} \Delta_{245} \\ \Delta_{125} \Delta_{236} = \Delta_{126} \Delta_{235} + \Delta_{123} \Delta_{256} \\ \Delta_{235} \Delta_{124} = \Delta_{123} \Delta_{245} + \Delta_{234} \Delta_{125} \end{array} \quad \begin{array}{l} \Delta_{245} \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix} \\ \Delta_{256} \\ \Delta_{125} \\ \Delta_{235} \\ \Delta_{123} \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ \Delta_{234} \\ \Delta_{345} \\ \Delta_{456} \\ \Delta_{156} \\ \Delta_{126} \end{array}$$

The topological realization of this cluster algebra, in which the marked surface (\mathbf{S}, \mathbf{M}) is a once-punctured quadrilateral, is shown in Figure 43. The initial cluster that we chose in (16.1) corresponds to the triangulation shown in Figure 44.

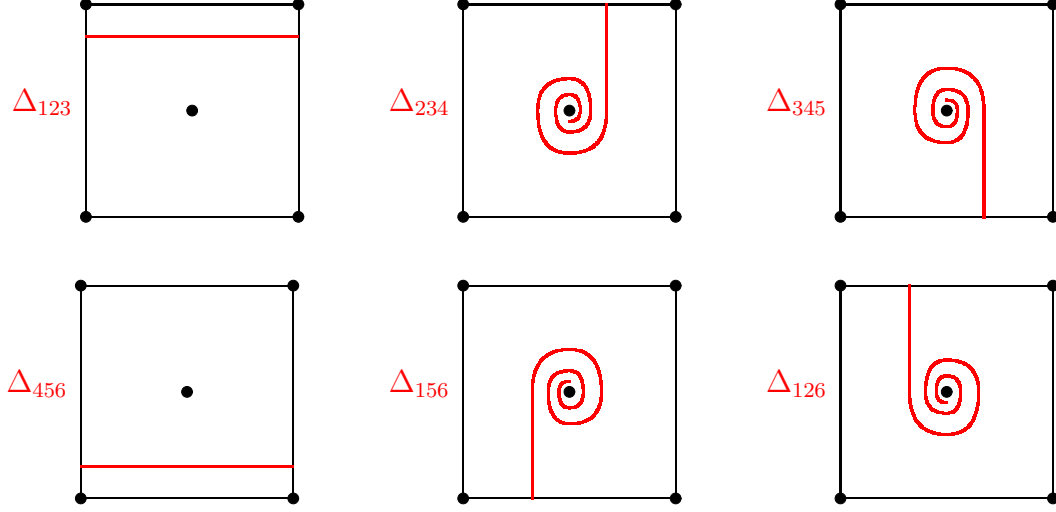


FIGURE 43. Representing the cluster structure on the Grassmannian $\text{Gr}_{3,6}$ by a multi-lamination \mathbf{L} on a once-punctured quadrilateral. Each of the 6 laminations in \mathbf{L} consists of a single curve, and corresponds to a particular coefficient variable (a Plücker coordinate).

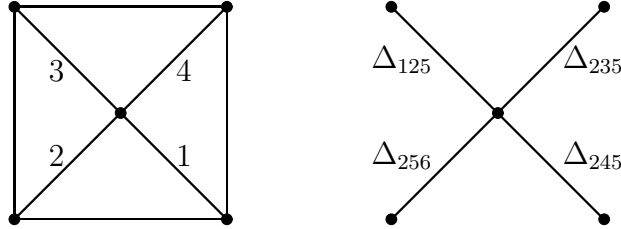


FIGURE 44. The triangulation representing the initial cluster (16.1) for the Grassmannian $\text{Gr}_{3,6}$. The remaining 12 tagged arcs (not shown in the picture) correspond to the rest of the cluster variables.

Example 16.4 (*The ring $\mathbb{C}[\text{Mat}_{3,3}]$*). The ring of polynomials in 9 variables z_{ij} ($i, j \in \{1, 2, 3\}$) viewed as matrix entries of a 3×3 matrix

$$z = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} \in \text{Mat}_{3,3} \cong \mathbb{C}^9$$

carries a natural cluster algebra structure of cluster type D_4 ; see, e.g., [10, 15, 30]. This cluster structure is very similar to the one discussed in Example 16.3, and is

obtained as follows. The map $\text{Mat}_{3,3} \rightarrow \text{Gr}_{3,6}$ defined by

$$z \mapsto \text{rowspan} \begin{pmatrix} z_{11} & z_{12} & z_{13} & 0 & 0 & 1 \\ z_{21} & z_{22} & z_{23} & 0 & -1 & 0 \\ z_{31} & z_{32} & z_{33} & 1 & 0 & 0 \end{pmatrix}$$

induces a ring homomorphism

$$\varphi : \mathbb{C}[\text{Gr}_{3,6}] \rightarrow \mathbb{C}[z_{11}, \dots, z_{33}].$$

For example, $\varphi(\Delta_{145}) = z_{11}$. More generally, φ maps the 19 Plücker coordinates Δ_{ijk} —all but Δ_{456} —into the 19 minors of z . In fact, φ sends each of the 16 cluster variables in $\mathbb{C}[\text{Gr}_{3,6}]$ into a cluster variable in $\mathbb{C}[z_{11}, \dots, z_{33}]$, and sends all but one coefficient variables in $\mathbb{C}[\text{Gr}_{3,6}]$ into coefficient variables in $\mathbb{C}[z_{11}, \dots, z_{33}]$, the sole exception being $\varphi(\Delta_{456}) = 1$. The exchange relations in the cluster algebra $\mathbb{C}[z_{11}, \dots, z_{33}]$ are obtained from those in $\mathbb{C}[\text{Gr}_{3,6}]$ by applying φ . As a result, we get a cluster structure that can be realized by a multi-lamination on a once-punctured quadrilateral that consists of 5 laminations: the ones shown in Figure 43 with the exception of the leftmost lamination in the bottom row.

Example 16.5 (*The special linear group SL_3 ; cf. [10]*). This cluster algebra (also of cluster type D_4) is obtained from the one in Example 16.4 by further specializing the coefficients, this time setting $\det(z) = 1$. Equivalently, we send both coefficient variables Δ_{123} and Δ_{456} in Example 16.3 to 1. This translates into removing the leftmost laminations in each of the two rows in Figure 43.

Example 16.6 (*Affine base space SL_4/N*). This example—with SL_4 replaced by an arbitrary complex semisimple Lie group—was one of the main examples that motivated the introduction of cluster algebras [12]. Let N denote the subgroup of unipotent upper-triangular matrices in $\text{SL}_4(\mathbb{C})$. The group N acts on SL_4 by left multiplication; let

$$\mathcal{A} = \mathbb{C}[\text{SL}_4/N] \subset \mathbb{C}[z_{11}, \dots, z_{44}]/\langle \det(z) - 1 \rangle$$

be the ring of N -invariant polynomials in the matrix entries z_{ij} . The ring \mathcal{A} is generated by the *flag minors*

$$\Delta_I : z = (z_{ij}) \mapsto \det(z_{ij} | i \in I, j \leq |I|),$$

for $I \subsetneq \{1, 2, 3, 4\}$, $I \neq \emptyset$. These flag minors satisfy well-known Plücker-type relations.

The ring \mathcal{A} carries a natural cluster structure of type A_3 (see, e.g., [2, Section 2.6]), with 6 coefficient variables

$$(16.2) \quad \Delta_1, \Delta_{12}, \Delta_{123}, \Delta_4, \Delta_{34}, \Delta_{234},$$

and 9 cluster variables

$$(16.3) \quad \Delta_2, \Delta_3, \Delta_{13}, \Delta_{14}, \Delta_{23}, \Delta_{24}, \Delta_{124}, \Delta_{134}, \Omega = -\Delta_1 \Delta_{234} + \Delta_2 \Delta_{134}.$$

Let the initial cluster be $\mathbf{x} = (\Delta_2, \Delta_3, \Delta_{23})$. Then the exchange relations from the initial seed are:

$$\begin{aligned}\Delta_2\Delta_{13} &= \Delta_{12}\Delta_3 + \Delta_1\Delta_{23} \\ \Delta_3\Delta_{24} &= \Delta_4\Delta_{23} + \Delta_{34}\Delta_2 \\ \Delta_{23}\Omega &= \Delta_{123}\Delta_{34}\Delta_2 + \Delta_{12}\Delta_{234}\Delta_3\end{aligned}$$

This algebra can be described using the multi-lamination \mathbf{L} of a hexagon shown in Figure 45.

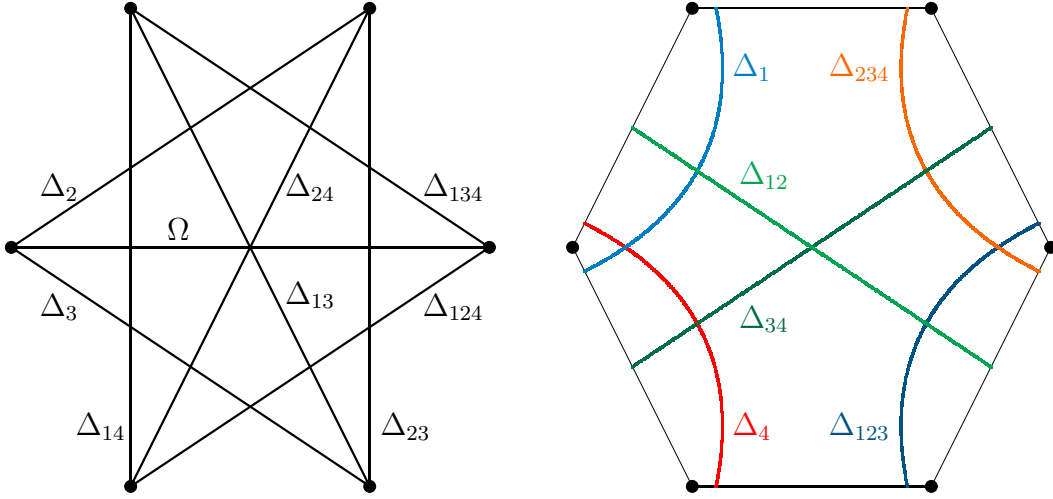


FIGURE 45. Representing the cluster structure on the affine base space SL_4/N by a multi-lamination of a hexagon. The 9 cluster variables (cf. (16.3)) correspond to the 9 arcs (diagonals) as shown on the left. The 6 coefficient variables (cf. (16.2)) correspond to the 6 single-curve laminations as shown on the right.

Remark 16.7. The cluster algebra of Example 16.6 illustrates a relatively rare phenomenon where the same algebra can be given two *different* topological realizations—because its exchange matrices can be obtained from two topologically different surfaces, in this case an unpunctured hexagon and a once-punctured triangle. (Informally speaking, types A_3 and D_3 coincide.) Consequently, one can choose to represent the cluster structure in $\mathbb{C}[\mathrm{SL}_4/N]$ by a multi-lamination in a once-punctured triangle. Cf. [9, Examples 4.3 and 4.5] and the discussion at the beginning of [9, Section 14].

Example 16.8 (*Unipotent subgroup $N_- \subset \mathrm{SL}_4$*). The coordinate ring of the subgroup N_- of unipotent lower-triangular 4×4 matrices carries a cluster algebra structure that can be obtained by specialization of coefficients from the cluster structure in $\mathbb{C}[\mathrm{SL}_4/N]$ discussed in Example 16.6: we set $\Delta_1 = \Delta_{12} = \Delta_{123} = 1$. (Cf. [2] and [19, Sections 4.2.6–4.3].) This corresponds to removing the three laminations with these labels from Figure 45.

Example 16.9 (*Affine base space SL_5/N and unipotent subgroup $N_- \subset \mathrm{SL}_5$*). In these two cases, one gets cluster algebras of type D_6 , which can be represented by a collection of 8 (respectively, 4) single-curve laminations in a once-punctured hexagon. We omit the details for the sake of brevity.

Further examples of similar nature include various coordinate rings discussed in [9, Examples 6.7, 6.9, 6.10], among others.

17. PRINCIPAL AND UNIVERSAL COEFFICIENTS

In this section, we work out two particular cases of our main construction, yielding two distinguished choices of coefficients introduced in [14].

Definition 17.1 (*Principal coefficients* [14, Definition 3.1, Remark 3.2]). Let $(\Sigma_t) = (\mathbf{x}(t), \tilde{B}(t))$ be an exchange pattern of geometric type. We say that this pattern (or the corresponding cluster algebra \mathcal{A}) has *principal coefficients* with respect to an initial vertex t_0 if the extended exchange matrix $\tilde{B}(t_0)$ has order $2n \times n$ (as always, n is the rank of the pattern), and the bottom $n \times n$ part of $\tilde{B}(t_0)$ is the identity matrix. We denote $\mathcal{A} = \mathcal{A}_\bullet(B(t_0))$, where $B(t_0)$ is the initial $n \times n$ exchange matrix.

To rephrase, let $\mathbf{x}(t_0) = (x_1, \dots, x_n)$ be the initial cluster, and let $B(t_0) = (b_{ij})$. Then the algebra $\mathcal{A} = \mathcal{A}_\bullet(B(t_0))$ is defined as follows. The ground semifield of \mathcal{A} is $\mathrm{Trop}(q_1, \dots, q_n)$, and the exchange relations (2.9) out of the initial seed Σ_{t_0} are

$$x_k x'_k = q_k \prod_{\substack{1 \leq i \leq n \\ b_{ik} > 0}} x_i^{b_{ik}} + \prod_{\substack{1 \leq i \leq n \\ b_{ik} < 0}} x_i^{-b_{ik}}.$$

To construct a topological model for a cluster algebra with principal coefficients, we will need the following notion.

Definition 17.2 (*Elementary lamination associated with a tagged arc*). Let $\gamma \in \mathbf{A}^\circ(\mathbf{S}, \mathbf{M})$ be a tagged arc in (\mathbf{S}, \mathbf{M}) . We denote by L_γ a lamination consisting of a single curve (also denoted by L_γ , by an abuse of notation) defined as follows (up to isotopy). The curve L_γ runs along γ in a small neighborhood thereof; to complete its description, we only need to specify what happens near the ends. Assume that γ has an endpoint a on the boundary of \mathbf{S} , more specifically on a circular component C . Then L_γ begins at a point $a' \in C$ located near a in the counterclockwise direction, and proceeds along γ as shown in Figure 46 on the left.

If γ has an endpoint $a \in \overline{\mathbf{M}}$ (a puncture), then L_γ spirals into a : counterclockwise if γ is tagged plain at a , and clockwise if it is notched. See Figure 46 on the right.

The following statement is straightforward to check.

Proposition 17.3. *Let T be a tagged triangulation with a signed adjacency matrix $B(T)$. Then $\mathcal{A}_\bullet(B(T)) \cong \mathcal{A}(\mathbf{S}, \mathbf{M}, \mathbf{L}_T)$ (cf. Definition 15.7), where $\mathbf{L}_T = (L_\gamma)_{\gamma \in T}$ is the multi-lamination consisting of elementary laminations associated with the tagged arcs in T .*

Example 17.4. Let $T = (\gamma_{ap}, \gamma_{bp})$ be the triangulation of a once-punctured digon shown in Figure 38. Then $\mathbf{L}_T = (L_4, L_3)$, in the notation of Figure 36.

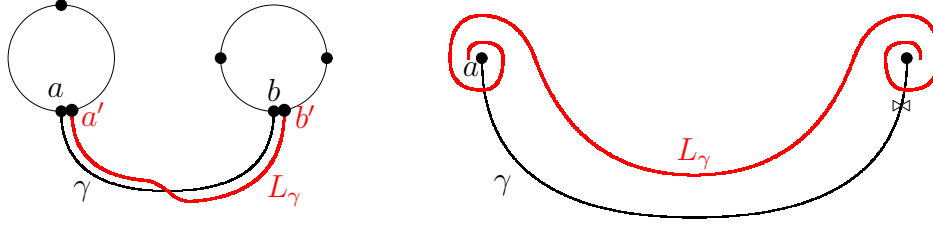


FIGURE 46. Elementary laminations associated with tagged arcs

We next turn to the discussion of cluster algebras with universal coefficients, which were constructed in [14, Section 12] for any finite Cartan-Killing (cluster) type. More explicit versions of this construction for the types A and D have been given in the unpublished work [17]; the type A case was reproduced in [33, Proposition 7.2]. Here we provide a restatement of the aforementioned results for the types A and D .

Proposition 17.5. *Let (\mathbf{S}, \mathbf{M}) be a marked surface of finite cluster type, so that the set of tagged arcs $\mathbf{A}^\bowtie(\mathbf{S}, \mathbf{M})$ is finite. The corresponding cluster algebra with universal coefficients can be realized as $\mathcal{A}(\mathbf{S}, \mathbf{M}, \mathbf{L}_\circ)$, where the multi-lamination \mathbf{L}_\circ consists of all elementary laminations (see Definition 17.2) associated with the tagged arcs $\gamma \in \mathbf{A}^\bowtie(\mathbf{S}, \mathbf{M})$.*

The straightforward proof of this proposition is omitted.

We note that among simply-laced finite cluster types, only types A and D , and their direct products, have topological realizations. (see [9, Examples 6.6–6.7, Remark 13.5]).

Example 17.6. Let (\mathbf{S}, \mathbf{M}) be the once-punctured digon, so that the cluster type is $A_1 \times A_1$. Then $\mathbf{L}_\circ = (L_1, L_3, L_4, L_6)$, in the notation of Figure 36.

Example 17.7. Let (\mathbf{S}, \mathbf{M}) be an unpunctured $(n + 3)$ -gon (type A_n). Then \mathbf{L}_\circ consists of all elementary (i.e., single-curve) laminations in (\mathbf{S}, \mathbf{M}) . These are the $\frac{n(n+3)}{2}$ curves connecting non-adjacent midpoints of the sides of the polygon.

Example 17.8. Let (\mathbf{S}, \mathbf{M}) be a once-punctured n -gon (type D_n). Again, \mathbf{L}_\circ consists of all elementary laminations in (\mathbf{S}, \mathbf{M}) . They include: $(n - 3)n$ curves connecting non-adjacent midpoints as in the previous example (the number is doubled since we can go on either side of the puncture); plus n curves connecting adjacent midpoints, going around the puncture; plus $2n$ curves starting at one of the n midpoints and spiraling into the puncture (either clockwise or counterclockwise); for the grand total of n^2 .

APPENDIX A. TROPICAL DEGENERATION AND RELATIVE LAMBDA LENGTHS

Here we sketch how our main construction of exchange patterns formed by generalized lambda lengths in “laminated Teichmüller spaces” can be obtained by *tropical degeneration* from a somewhat more traditional construction in which laminations are replaced by hyperbolic structures of ordinary kind.

First, a simple observation concerning (commutative) semifields; see Definition 3.3.

Lemma A.1. *A direct product of semifields is a semifield, under component-wise operations.*

The tropical semifield of Definition 4.1 is one such example:

$$(A.1) \quad \text{Trop}(q_1, q_2, \dots) \cong \text{Trop}(q_1) \times \text{Trop}(q_2) \times \dots$$

We next describe, somewhat informally, the general procedure of tropical degeneration. Fix a real number k , and define the binary operation \oplus_k by

$$(A.2) \quad y \oplus_k z = (y^k + z^k)^{1/k},$$

where y and z are positive reals, or positive-valued functions on some fixed set. It is easy to see that the binary operation \oplus_k is commutative, associative, and distributive with respect to the ordinary multiplication. This makes $(\mathbb{P}, \oplus_k, \cdot)$ a semifield, where \mathbb{P} is any set of numbers or functions that is closed under \oplus_k and under multiplication. The simplest instance of this construction produces the semifield $(\mathbb{R}_{>0}, \oplus_k, \cdot)$ of positive real numbers, with ordinary multiplication and with addition \oplus_k defined by (A.2). In the “tropical limit,” as $k \rightarrow -\infty$, we get the semifield $(\mathbb{R}_{>0}, \oplus, \cdot)$ with the addition \oplus given by

$$(A.3) \quad y \oplus z = \lim_{k \rightarrow -\infty} y \oplus_k z = \min(y, z).$$

This is a close relative of the tropical addition (4.2), in the one-dimensional version, with the set I in (4.1) consisting of a single element. To obtain a multi-dimensional version, we apply the direct product construction of Lemma A.1, either after taking the limit (as in (A.1)), or before. Let us discuss the latter approach in more concrete detail. Start with a collection of real parameters $\mathbf{k} = (k_i)_{i \in I}$, and consider the semifield of tuples $\mathbf{y} = (y_i)_{i \in I}$ with the ordinary (pointwise) multiplication and with the addition $\oplus_{\mathbf{k}}$ defined by

$$(\mathbf{y} \oplus_{\mathbf{k}} \mathbf{z})_i = y_i \oplus_{k_i} z_i = (y_i^{k_i} + z_i^{k_i})^{1/k_i}.$$

Taking the limits $k_i \rightarrow -\infty$ for all i , we obtain a close relative of the tropical semifield (4.1), with the addition defined by

$$(\mathbf{y} \oplus \mathbf{z})_i = \min(y_i, z_i).$$

The geometric counterpart of the tropical degeneration procedure consists in obtaining a lamination, viewed as a point on the Thurston boundary [4, 32] of the appropriate Teichmüller space, as a limit of ordinary hyperbolic structures. For simplicity, we discuss the case of surfaces with no marked points in the interior. The general case can be handled in exactly the same way as before, by lifting arcs to the opened surface. We still want to get the most general coefficients, so we cannot use the construction of Theorem 8.6 (coefficients coming from boundary segments); rather, we shall aim at obtaining the construction of Theorem 15.6 (coefficients coming from a multi-amination) in the restricted generality of surfaces with no punctures.

Let $\sigma \in \widetilde{\mathcal{T}}(\mathbf{S}, \mathbf{M})$ be a decorated hyperbolic structure. (In the presence of punctures, we would need to consider $\sigma \in \overline{\mathcal{T}}(\mathbf{S}, \mathbf{M})$.) The lambda lengths $\lambda_\sigma(\gamma)$ satisfy generalized exchange/Ptolemy relations (7.2):

$$(A.4) \quad \lambda_\sigma(\eta)\lambda_\sigma(\theta) = \lambda_\sigma(\alpha)\lambda_\sigma(\gamma) + \lambda_\sigma(\beta)\lambda_\sigma(\delta).$$

Choose $k \in \mathbb{R}$, and set

$$\lambda_{\sigma,k}(\gamma) = (\lambda_\sigma(\gamma))^{1/k}.$$

These quantities satisfy the \oplus_k -version of (A.4):

$$(A.5) \quad \lambda_\sigma(\eta)\lambda_\sigma(\theta) = \lambda_\sigma(\alpha)\lambda_\sigma(\gamma) \oplus_k \lambda_\sigma(\beta)\lambda_\sigma(\delta).$$

Recalling Theorem 7.4, pick a triangulation T and a collection of positive reals $(c(\gamma))_{\gamma \in T \cup \mathbf{B}(\mathbf{S}, \mathbf{M})}$. Then let σ and k vary so that $k \rightarrow -\infty$ while the coordinates $(\lambda_{\sigma,k}(\gamma))$ remain fixed:

$$\lambda_{\sigma,k}(\gamma) = c(\gamma), \quad \text{for } \gamma \in T \cup \mathbf{B}(\mathbf{S}, \mathbf{M}).$$

In other words, we set $\lambda_\sigma(\gamma) = c(\gamma)^k$ and let $k \rightarrow -\infty$. As a result, if all of the $c(\gamma) < 1$, then the λ -coordinates will go to $+\infty$ and σ goes to a point on the Thurston boundary of $\tilde{\mathcal{T}}(\mathbf{S}, \mathbf{M})$ which can be identified with a real measured lamination L whose tropical lambda lengths match the values we picked:

$$c_L(\gamma) = c(\gamma), \quad \text{for } \gamma \in T \cup \mathbf{B}(\mathbf{S}, \mathbf{M}).$$

Meanwhile, the exchange relations (A.5) degenerate into

$$c_L(\eta)c_L(\theta) = c_L(\alpha)c_L(\gamma) \oplus c_L(\beta)c_L(\delta),$$

where \oplus denotes the minimum (the tropical addition). This fairly standard argument shows how the tropical exchange relations for the tropical lambda lengths associated with a single lamination can be obtained by tropical degeneration from the ordinary exchange relations for the lambda lengths with respect to a hyperbolic structure.

(If some of the $c(\gamma) > 1$, then some λ -coordinates will go to 0. This can still be interpreted as σ going to a point on a generalized “Thurston boundary” of Teichmüller space, but is more confusing geometrically.)

To extend this construction to the case of multiple laminations, we can use the direct product construction of Lemma A.1, yielding a solution of exchange relations in the tropical semifield with several parameters, as in Lemmas 14.9 and 14.11. This solution can then be used to renormalize the ordinary lambda lengths as in (15.2), delivering the requisite normalized exchange pattern by virtue of Proposition 3.4, as in the proof of Theorem 15.6.

As we have seen, the process of creating a normalized exchange pattern formed by renormalized lambda lengths consists of three main stages:

- (1) producing a tropical solution of exchange relations from an ordinary one by means of “tropical degeneration”;
- (2) applying a direct product construction to get a solution depending on several reference geometries; and
- (3) dividing the original non-normalized solution by the one produced at stage (2) to obtain a normalized pattern.

Alternatively, stage (1) (tropical degeneration) can be executed after stage (2) or even after stage (3). To conclude this appendix, we show how to perform stage (3) in the restricted case of a single reference geometry (where stage (2) is trivial). To compensate for this restriction, we will consider a general case of a bordered surface *with* punctures. This construction bears some similarity to the work of V. Fock and

A. Goncharov on cluster varieties, and in particular to their notion of *symplectic double* [8, Section 2.2].

Fix a reference geometry, i.e., a decorated hyperbolic structure $\varrho \in \overline{\mathcal{T}}(\mathbf{S}, \mathbf{M})$. Now, for any other $\sigma \in \overline{\mathcal{T}}(\mathbf{S}, \mathbf{M})$ satisfying the boundary conditions

$$(A.6) \quad \lambda_\sigma(\beta) = \lambda_\varrho(\beta) \text{ for all } \beta \in \mathbf{B}(\mathbf{S}, \mathbf{M}),$$

$$(A.7) \quad \lambda_\sigma(p) = \lambda_\varrho(p) \text{ for all } p \in \overline{\mathbf{M}},$$

the *relative lambda length* $\lambda_{\sigma/\varrho}(\gamma)$ of an arc $\gamma \in \mathbf{A}^\circ(\mathbf{S}, \mathbf{M})$ is defined by

$$(A.8) \quad \lambda_{\sigma/\varrho}(\gamma) = \lambda_\sigma(\overline{\gamma}) / \lambda_\varrho(\overline{\gamma}),$$

where $\overline{\gamma}$ is an arbitrary lift of γ to the opened surface $(\overline{\mathbf{S}}, \overline{\mathbf{M}})$. Note that this ratio does not depend on the choice of a lift $\overline{\gamma}$ since the numerator and the denominator rescale by the same factor under the twists ψ_p (see (10.12)).

By Theorem 11.1, the lambda lengths $\lambda_\varrho(\gamma)$ (respectively, $\lambda_\sigma(\gamma)$) form a non-normalized exchange pattern with coefficients in the multiplicative group generated by the boundary parameters (A.6)–(A.7). Consequently, by Proposition 3.4, the relative lambda lengths $\lambda_{\sigma/\varrho}(\gamma)$ form a normalized exchange pattern over the semifield generated by the lambda lengths for ϱ , with ordinary addition and multiplication. To illustrate what this amounts to, consider a quadrilateral in (\mathbf{S}, \mathbf{M}) with sides $\alpha, \beta, \gamma, \delta$ and diagonals η and θ , as in Proposition 7.6. Let $\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}$ be lifts of $\alpha, \beta, \gamma, \delta$ to the opened surface $(\overline{\mathbf{S}}, \overline{\mathbf{M}})$, coordinated so that $\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}$ still form a quadrilateral. (Cf. the discussion surrounding (15.4).) Define the cross-ratio τ of the edge ϱ by

$$(A.9) \quad \tau = \frac{\lambda_\varrho(\overline{\alpha})\lambda_\varrho(\overline{\gamma})}{\lambda_\varrho(\overline{\beta})\lambda_\varrho(\overline{\delta})};$$

again, the value of τ does not depend on the choice of a quadruple of coordinated lifts $\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}$. Then

$$(A.10) \quad \lambda_{\sigma/\varrho}(\eta)\lambda_{\sigma/\varrho}(\theta) = \frac{\tau}{1+\tau}\lambda_{\sigma/\varrho}(\alpha)\lambda_{\sigma/\varrho}(\gamma) + \frac{1}{1+\tau}\lambda_{\sigma/\varrho}(\beta)\lambda_{\sigma/\varrho}(\delta).$$

This exchange relation can of course be obtained by dividing the corresponding Ptolemy relations for the two sets of lambda lengths (*viz.*, λ_σ and λ_ϱ) by one another. Furthermore, the coefficients $\frac{\tau}{1+\tau}$ and $\frac{1}{1+\tau}$ (or their ratio τ , in the “Y-pattern version” [14]) transform under flips according to the appropriate mutation rules, as predicted by Propositions 3.1–3.4. (In the case of ordinary flips on (\mathbf{S}, \mathbf{M}) , this observation was already made in [5, 6, 21].)

Degenerating the hyperbolic structure ϱ to a lift \overline{L} of a lamination L as explained in the first part of this section, we obtain a normalized exchange pattern of geometric type. More concretely, we get

$$\frac{\tau}{1+\tau} = \frac{\lambda_\varrho(\overline{\alpha})\lambda_\varrho(\overline{\gamma})}{\lambda_\varrho(\overline{\eta})\lambda_\varrho(\overline{\theta})} \rightarrow \frac{c_{\overline{L}}(\overline{\alpha})c_{\overline{L}}(\overline{\gamma})}{c_{\overline{L}}(\overline{\eta})c_{\overline{L}}(\overline{\theta})} \quad \text{and} \quad \frac{1}{1+\tau} = \frac{\lambda_\varrho(\overline{\beta})\lambda_\varrho(\overline{\delta})}{\lambda_\varrho(\overline{\eta})\lambda_\varrho(\overline{\theta})} \rightarrow \frac{c_{\overline{L}}(\overline{\beta})c_{\overline{L}}(\overline{\delta})}{c_{\overline{L}}(\overline{\eta})c_{\overline{L}}(\overline{\theta})},$$

recovering the coefficients (15.4). It is also easy to see directly that these coefficients are obtained by exponentiating the shear coordinates of L , conforming to our general recipe.

APPENDIX B. VERSIONS OF TEICHMÜLLER SPACES AND COORDINATES

This appendix provides a brief guide to help the reader navigate among various spaces related to the Teichmüller space which appear throughout this paper, and among different coordinate systems on these spaces. There are several independent choices involved in these constructions. All of them refer to various structures that can be put on a compact surface \mathbf{S} , possibly with boundary, with a suitable finite set of marked points \mathbf{M} ; these points may lie in the interior (*punctures*) or on the boundary.

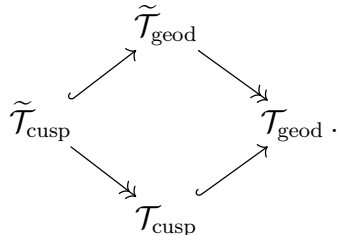
The first choice is between *geometric* and *tropical* coordinates. Geometric coordinates, for instance the exponentiated length of a curve, are coordinates on a Teichmüller space; they satisfy algebraic relations like the Ptolemy relation (7.2). Tropical coordinates, on the other hand, are coordinates on a space of *measured laminations*, which for us amount to collections of non-intersecting curves, with some restrictions. Such a measured lamination L can be thought of as defining a degenerate metric for which all contributions to distances come from crossing the lamination L . Thus tropical coordinates are some variations of intersection numbers. The relations they satisfy are naturally written in terms of operations in an appropriate tropical semifield (see Definition 4.1).

The next choice is between *cusped* and *opened* surfaces. Geometrically, a hyperbolic cusped surface has an infinitely long “horn” of finite area at each of the punctures in the interior of \mathbf{S} , while an opened surface may have a geodesic boundary opened up at these same punctures. We also need to orient the boundary at these openings, as in Definition 10.2. In the tropical world, laminations on a cusped surface avoid the punctures, while on an opened surface, they are allowed to spiral around the opened boundary.

We then consider some version of *lambda lengths* or *shear coordinates*. Lambda lengths are in principle simpler, being essentially the exponentiated lengths of arcs connecting marked points. Shear coordinates, on the other hand, depend on an arc in the context of a particular triangulation. Lambda lengths are coordinates on *decorated* Teichmüller spaces, involving some choice for each marked point, while shear coordinates do not need this choice.

Finally, each set of coordinates can be extended to work with *tagged* arcs or triangulations; this is necessary to complete the cluster algebra structure.

The choices for geometric coordinates (ignoring the tagging) can be summarized by the following diagram:



Here $\mathcal{T}_{\text{cusp}}$ is the Teichmüller space of cusped surfaces, while $\mathcal{T}_{\text{geod}}$ is the space of opened surfaces with geodesic boundary. In both cases, $\tilde{\mathcal{T}}$ is the decorated Teichmüller space, which allows the definition of lambda lengths, and the coordinates on the undecorated space \mathcal{T} are shear coordinates.

While each combination of these choices makes sense, we do not explicitly describe them all in this paper. Two natural choices are lambda lengths for decorated cusped surfaces ($\tilde{\mathcal{T}}_{\text{cusp}}$ above) and shear coordinates for opened surfaces ($\mathcal{T}_{\text{geod}}$ above). These correspond to the \mathcal{A} and \mathcal{X} spaces of Fock and Goncharov [5], respectively. Shear coordinates on $\mathcal{T}_{\text{cusp}}$ are not independent—they satisfy a relation for each puncture. From the geometric point of view, the principal novelty of this paper is to consider lambda lengths for $\tilde{\mathcal{T}}_{\text{geod}}$ as explained in Section 10.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR, MI 48109, USA
E-mail address: fomin@umich.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720, USA
E-mail address: dpt@math.berkeley.edu